

Unit 10

# Dynamics



# Introduction

In this unit you will learn how to use mathematics to describe and predict the motion of objects. In Unit 5, where you studied *statics*, you saw how to use Newton's first and third laws of motion to solve problems involving objects at rest. There, the forces acting on an object were in equilibrium. In this unit, you will study situations where the forces acting on an object are not in equilibrium, so that the sum of the forces (the resultant force) acting on the object is non-zero. You will see how you can use Newton's second law to determine the motion that results from such forces. The area of applied mathematics concerned with using Newton's laws to study the motion of objects is called **Newtonian dynamics**, or just **dynamics**.

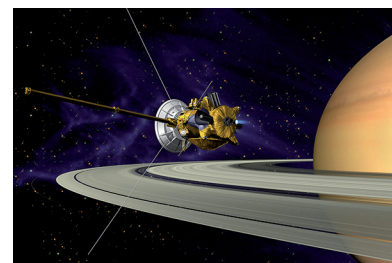
The methods that you will study in this unit are used by mathematicians, physicists and engineers to solve a wide range of practical problems, from problems involving the motion of vehicles and the paths taken by balls thrown through the air, to problems involving the movement of planets and other objects through space. For example, in 2004 the Cassini–Huygens spacecraft was made to perform a complicated set of calculated manoeuvres to get it in the correct orbit around the planet Saturn.

As was mentioned in the introduction to Unit 5, Newton's laws provide a mathematical model of situations in the real world that allows us to apply mathematical methods to these situations. However, there are situations for which the model provided by Newton's laws is not appropriate, such as those involving extremely small objects (where the laws of quantum mechanics come into play) or objects moving extremely fast (where relativistic effects need to be incorporated).

We will start our study of dynamics, in Sections 1 and 2 of this unit, by considering motion that is along a straight line. In Section 1 you will see how such motion can be described mathematically, and work with the relationships between the position of an object, its velocity and its acceleration. In Section 2 you will see how to use Newton's second law to predict the motion of objects under the influence of forces.

Finally, in Section 3 you will see how the methods of Sections 1 and 2 can be generalised to apply to motion in two or three dimensions.

This unit uses the techniques of vectors, and of differentiation and integration, that you studied in MST124, and have already used in some earlier units in this module.



The Cassini–Huygens mission to Saturn

Using Newton's dynamics, mathematicians in the 18th century developed the new science of celestial mechanics – the term was coined by the French mathematician Pierre-Simon Laplace (1749–1827). The aim was to show that Newton's theory could account for all the motions in the heavens.

One of the hardest problems was the motion of the moon – it was so difficult that even Newton confessed that it made his head ache! This problem was important if a method was to be found for assessing longitude when at sea. Among those who made great strides towards a solution was the Swiss mathematician Leonhard Euler (1707–83), who was awarded £300 by the British Board of Longitude for theorems used in the construction of lunar tables.

In the 19th century, one of the greatest triumphs of Newtonian dynamics was the discovery in 1846 of the planet Neptune, which was predicted mathematically before it was observed by both the French astronomer Urbain Le Verrier (1811–77) and the British mathematician John Couch Adams (1819–92). The question of who deserved the credit for the discovery created quite some controversy.

## 1 Motion along a straight line

The motion of a real object is often complicated and difficult to describe. Consider, for example, a leaf falling to the ground. Rather than falling straight down, the leaf will probably follow a complicated three-dimensional path. The path may be affected by other objects, such as other falling leaves. While falling, the leaf might rotate, or bend.

Rather than try to describe such complicated motion mathematically, in this unit we will simplify the problems that we consider by making a number of *modelling assumptions*, as we did in Unit 5. Our main modelling assumptions are as follows.

First, we will model all moving objects as particles. As you saw in Unit 5, a particle has mass but no size. By modelling an object as a particle, we can ignore any rotation or change of shape of the object.

Second, we will always consider the motion of only a single object. So we will ignore any interactions between objects.

Finally, in this section and in Section 2 we will simplify things further by only considering problems where the motion occurs along a straight line. Such motion is known as one-dimensional motion.

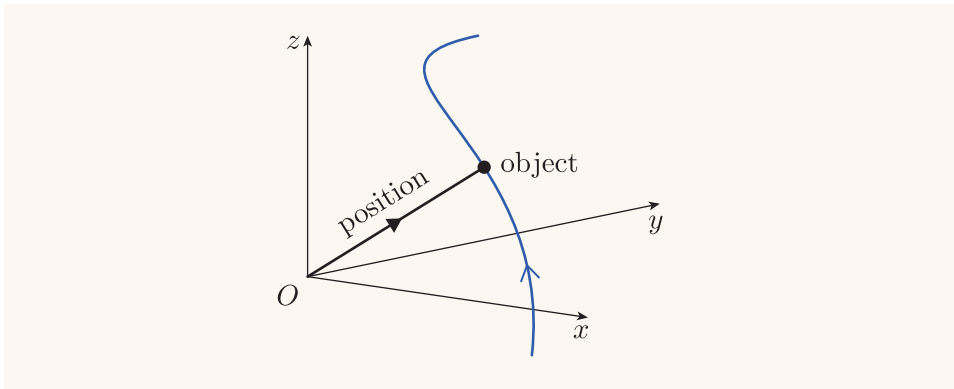
In this section you will see how you can describe and predict the motion of an object along a straight line by using the relationships between the position, velocity and acceleration of the object. This topic is known as **kinematics**; this term refers to the study of motion without consideration of the causes of the motion. In the next section you will see how you can relate the motion of an object to the forces that act on the object.



## 1.1 Position, velocity and acceleration

In this first subsection you will revise the ideas about the displacement, velocity and acceleration of an object moving along a straight line that you met in MST124 Units 6 and 7, and learn about some terminology and conventions that were not used in MST124, but are commonly used in dynamics.

When we want to model the motion of a moving object, we set up a coordinate system, and represent the position of the object at any instant in time by the *position vector* of the object at that instant – in other words, by the displacement of the object from the origin. We refer to this position vector simply as the **position** of the object. This is illustrated in Figure 1.



**Figure 1** The position of an object moving along a path

Then the **velocity** of the object at any instant in time is the rate of change of its position with respect to time at that instant, and the **acceleration** of the object at any instant in time is the rate of change of its velocity with respect to time at that instant. The position, velocity and acceleration of an object are all vector quantities and so have both a magnitude and a direction. In particular, the magnitude of the velocity of a moving object at any instant in time is the speed with which the object is moving at that instant, and the direction of the velocity is the direction in which the object is moving at that instant.

The position, velocity and acceleration of a moving object can all change with time, so each of these quantities is a *vector function* of time.

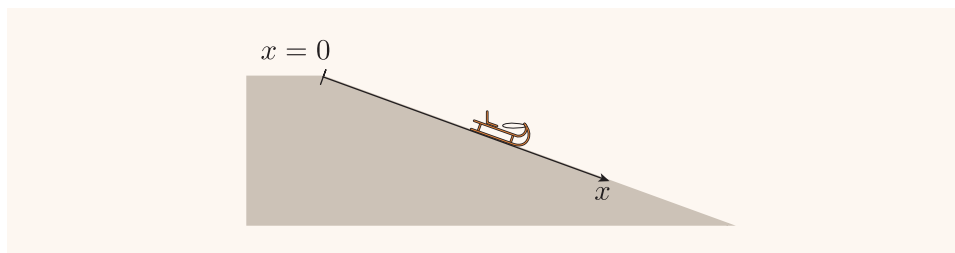
A **vector function** is a function whose output values are vectors. (Its input values can be scalars, like times, or vectors.)

In SI units, position is measured in metres and time is measured in seconds. The corresponding units for velocity are metres per second (written as  $\text{m/s}$  or  $\text{ms}^{-1}$ ), and the corresponding units for acceleration are metres per second squared (usually written as  $\text{m/s}^2$  or  $\text{ms}^{-2}$ ).

For an object that moves along a straight line, even though its position, velocity and acceleration are vector quantities, we can represent these quantities by scalars, with direction indicated by a plus or minus sign. Here is how we do this.

Since the motion is along a straight line, we need only a single coordinate axis, which we refer to as the  $x$ -axis. We choose this axis to lie along the line of motion, and we choose the origin to be some convenient point on the line of motion. If the object moves along the line in only one direction, then we usually choose the positive direction of the  $x$ -axis to be the direction of motion of the object, for convenience.

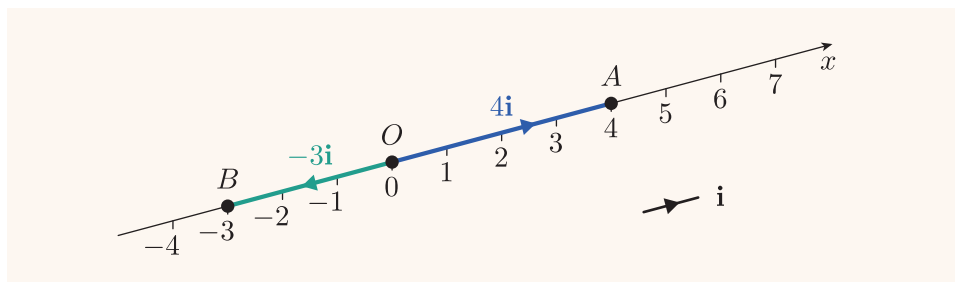
For example, consider a sledge that slides in a straight line down a slope, as illustrated in Figure 2.



**Figure 2** A sledge sliding down a slope, and a suitable  $x$ -axis and origin for this motion

We choose the  $x$ -axis to lie along the line of motion of the sledge, with the positive direction of the  $x$ -axis in the direction of motion of the sledge, which is down the slope. It makes sense to choose the origin to be at the top of the slope, as shown by the label  $x = 0$  in Figure 2.

Once we have chosen an  $x$ -axis in this way for an object moving along a straight line, the position of the object at any moment in time will be a vector of the form  $x\mathbf{i}$ , where, as usual,  $\mathbf{i}$  is the Cartesian unit vector in the positive direction of the  $x$ -axis. For example, for an object moving along the line in Figure 3, if the object is at point  $A$ , then its position is  $4\mathbf{i}$ , and if it is at point  $B$ , then its position is  $-3\mathbf{i}$ . Since every position along the line is a multiple of  $\mathbf{i}$  in this way, for convenience we usually omit the  $\mathbf{i}$ , and represent a position  $x\mathbf{i}$  simply by the scalar  $x$ . For example, in Figure 3, we represent the position of an object at point  $A$  by the scalar 4, and the position of an object at point  $B$  by the scalar  $-3$ . (For a practical situation, such as the sledge example, these scalars would include units.)



**Figure 3** Two positions along a straight line

So, when we are dealing with motion along a straight line, we will treat position as a scalar quantity, with direction from the origin indicated by the sign (plus or minus) of the scalar. You met this idea in MST124

Unit 6, though there the position of an object – in other words, its displacement from the origin – was referred to simply as its *displacement*.

For an object moving along a straight line, we usually use the letter  $x$  to denote its position, and  $t$  to denote time. The position of the object changes with time, so  $x$  is a function of  $t$ . For example, the position  $x$  of the sledge in Figure 2 might be given in terms of the time  $t$  by the function

$$x = 2t^3 - t^2.$$

Although we will usually denote the position of an object at time  $t$  simply by  $x$ , rather than by the function notation  $x(t)$ , we will sometimes use the notation  $x(t)$  when it is convenient.

Since we are representing the position of an object moving along a straight line as a scalar function of time, the velocity of the object (the rate of change of its position) and the acceleration of the object (the rate of change of its velocity) are also scalar functions of time, with direction indicated by the sign of the scalar in the same way as for position. We usually denote the velocity and acceleration of an object moving along a straight line by  $v$  and  $a$ , respectively (and occasionally by the function notation  $v(t)$  and  $a(t)$ ). So we have the following facts, which you met in MST124 Unit 6.

### Determining velocity and acceleration from position

If a particle moves along a straight line, with position  $x$ , velocity  $v$  and acceleration  $a$  at time  $t$ , then

$$v = \frac{dx}{dt} \quad \text{and} \quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

If you have an equation that gives the position of an object along a straight line in terms of time, then you can use the formulas in the box above to find the velocity and acceleration of the object in terms of time. For example, if the position of the sledge in Figure 2 is given by

$$x = 2t^3 - t^2,$$

then its velocity is given by

$$v = \frac{dx}{dt} = 6t^2 - 2t,$$

and its acceleration is given by

$$a = \frac{dv}{dt} = 12t - 2.$$

Since integration reverses differentiation, the following facts also hold.

### Determining position from velocity, and velocity from acceleration

If a particle moves along a straight line, with position  $x$ , velocity  $v$  and acceleration  $a$  at time  $t$ , then

$$x = \int v \, dt \quad \text{and} \quad v = \int a \, dt.$$

When you use integration to determine an equation for the position or velocity of an object in terms of time, as in the box above, the resulting equation will include a constant of integration. To find the value of this constant, you need another piece of information, such as the position or velocity of the object at a particular time. Such an additional piece of information is called an **initial condition**. You use it in the same way as you used an initial condition to find a value for a constant of integration in the general solution of a differential equation in Unit 8.

The next example and the activities that follow illustrate all the ideas that you have seen so far in this subsection. The problems here are similar to the problems about displacement, velocity and acceleration that you saw in MST124 Units 6 and 7.



A snowboarder sliding downhill

#### Example 1 *Determining acceleration and position from velocity*

A snowboarder slides down a straight slope, into a patch of wet snow. Her speed decreases from the time she hits the start of the wet snow, and eventually she comes to a halt. Her velocity  $v$  during the time that she is slowing down is modelled by the equation

$$v = 25 - \frac{t^2}{4},$$

where  $t$  is the time after she hits the wet snow. Time is measured in seconds, and velocity is measured in metres per second.

Find the following.

- The time when the snowboarder comes to a halt.
- Her acceleration at the time when she comes to a halt.
- Her position at the time when she comes to a halt, to three significant figures.

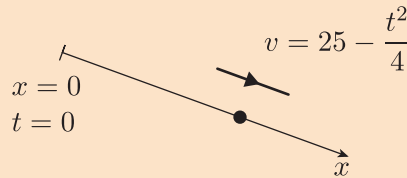
#### Solution

State any modelling assumptions.

Model the snowboarder as a particle.

☁ Draw a diagram summarising the situation. Represent the general position of the snowboarder as a dot. Represent the velocity as a vector. Choose the position of the  $x$ -axis, including where the origin is, and draw this axis on the diagram, labelling the origin with  $x = 0$ . Mark and label any other positions that may be useful: here mark the position of the snowboarder at time  $t = 0$ , when she is at the origin. ☁

Choose the  $x$ -axis to point down the slope, with the origin at the snowboarder's location at time  $t = 0$ , as shown in the diagram.



- (a) ☁ The snowboarder comes to a halt when her velocity is zero. ☁

The snowboarder comes to a halt when  $v = 0$ ; that is, when

$$25 - \frac{t^2}{4} = 0,$$

which gives

$$t^2 = 100.$$

Since the time  $t$  will be positive when the snowboarder comes to a halt, we obtain

$$t = 10.$$

The snowboarder comes to a halt 10 seconds after she hit the wet snow.

- (b) ☁ To find an expression for the acceleration, differentiate the expression for the velocity. ☁

The acceleration  $a$  of the snowboarder is given by

$$a = \frac{dv}{dt} = \frac{d}{dt} \left( 25 - \frac{t^2}{4} \right) = \frac{-2t}{4} = -\frac{t}{2}.$$

When  $t = 10$ , the acceleration is

$$a = -\frac{10}{2} = -5.$$

The acceleration of the snowboarder at the time when she comes to a halt is  $-5 \text{ m s}^{-2}$ .

- (c) To find an expression for the position, integrate the expression for the velocity.

The position  $x$  of the snowboarder is given by

$$x = \int v \, dt = \int \left( 25 - \frac{t^2}{4} \right) dt = 25t - \frac{t^3}{12} + c,$$

where  $c$  is a constant.

Find the value of  $c$  by using the initial condition that  $x = 0$  when  $t = 0$ .

Since  $x = 0$  when  $t = 0$ ,

$$0 = 25 \times 0 - \frac{0}{12} + c,$$

which gives  $c = 0$ .

The position of the snowboarder is therefore given by

$$x = 25t - \frac{t^3}{12}.$$

Her position at time  $t = 10$  is

$$x = 25 \times 10 - \frac{10^3}{12} = 166.66 \dots$$

The snowboarder comes to a halt when she is 167 m (to 3 s.f.) into the patch of wet snow.

Notice that in the example above the expression for the acceleration takes a negative value for all values of  $t$  involved in the problem. On the other hand, the expression for the velocity takes a positive value for all values of  $t$  involved – this is because the  $x$ -axis was chosen to point in the direction of motion. The fact that the acceleration and the velocity have opposite signs (that is, they point in opposite directions when they are considered as vectors) tells you that the snowboarder is slowing down.

In the next activity, the moving object, a parachutist, is moving vertically downwards, so you should take the positive direction of the  $x$ -axis to point vertically downwards, which makes the velocity positive. The question gives a formula for the *downwards* acceleration of the parachutist, and this formula always takes a negative value, so again in this activity the acceleration and the velocity have opposite signs. So the parachutist is slowing down, like the snowboarder.

When the acceleration and the velocity of an object moving along a straight line have the *same sign* (that is, they point in the same direction when they are considered as vectors), the moving object is speeding up.

Note that, as with all the formulas for position, velocity and acceleration that you will see in this unit, the formula for the velocity of the snowboarder in the example above and the formula for the acceleration of the parachutist in the activity below are just models – they may or may not be *good* models for what actually happens in real situations. Some of the models that you will meet in this unit are better than others!

Your first steps in solving the problem in the next activity should be to state any modelling assumptions and draw a diagram to summarise the situation, as in Example 1 above. You should start your solution to every dynamics problem in this way. You should draw a dot to represent the general position of the moving object. You should show the velocity by drawing and labelling a vector. You can show the acceleration in a similar way, but we usually use a vector with a double arrow for acceleration, as illustrated in Figure 4, to distinguish it from a velocity.



**Figure 4** A vector representing acceleration

You should choose an appropriate direction for the  $x$ -axis, and draw the  $x$ -axis on your diagram. Usually, you should mark the location of the object at time  $t = 0$ .

### Activity 1 Determining velocity and position from acceleration

The downwards acceleration of a parachutist, in  $\text{m s}^{-2}$ , is given by

$$a = -\frac{55}{2}e^{-\frac{5}{4}t}, \quad \text{for } t > 0,$$

where  $t$  is the time (in seconds) after the parachute opens. The downwards velocity of the parachutist at the time when the parachute opens is  $30 \text{ m s}^{-1}$ .

- Find the velocity of the parachutist in terms of the time  $t$  after the parachute opens.
- Find the position of the parachutist in terms of the time  $t$  after the parachute opens.



A parachute slows the parachutist's fall to a safe speed

In the next activity, the moving object moves along the line of motion in different directions at different times.

### Activity 2 *Determining position from velocity*



A shuttle carries the weft threads through the strung warp threads on a loom

To make cloth, a shuttle moves backwards and forwards across the full width of a loom. For a particular loom, the velocity, in  $\text{ms}^{-1}$ , of the shuttle is given in terms of the time  $t$  by

$$v = \frac{2}{\pi} \sin\left(\frac{\pi t}{2}\right),$$

where a positive velocity corresponds to motion to the right. The shuttle starts at the left-hand side of the loom at time  $t = 0$ .

- Find the position of the shuttle in terms of the time  $t$ .
- What is the width of the loom? Give your answer to the nearest centimetre.

Hint: the width of the loom is the difference between the maximum and minimum values of the position of the shuttle.

Finally in this subsection, here are a few remarks about some terminology used in dynamics.

As you were reminded earlier, the magnitude of the velocity of an object is the *speed* of the object. Notice also that the magnitude of the position of an object is its *distance* from the origin. Confusingly, the magnitude of acceleration is usually called acceleration! So the word ‘acceleration’ can mean either a vector quantity or a scalar quantity. The intended interpretation of the word should be clear from the context.

In everyday language, the word *deceleration* is often used to indicate a decreasing speed, and the word *acceleration* is reserved to indicate an increasing speed. However in mathematics and science the rate of change of the velocity of an object is always referred to as its acceleration, no matter whether the speed of the object is increasing or decreasing.

Finally, any equation relating two or more of the four quantities time, position, velocity and acceleration is known as an **equation of motion**. Many equations of motion involve derivatives, and so are differential equations. Much of this unit is devoted to using equations of motion.

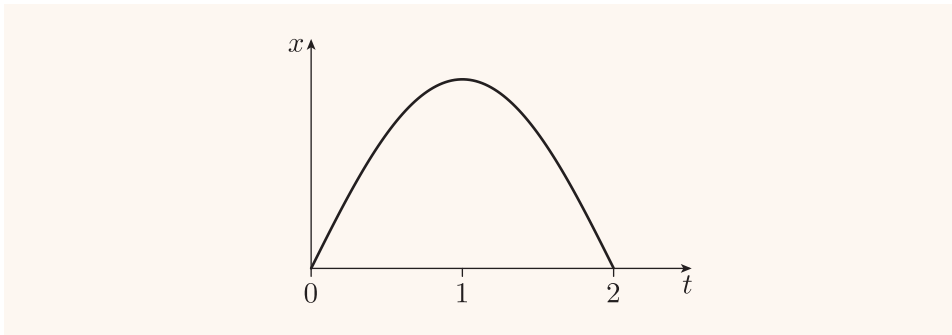


## 1.2 Graphs of position, velocity and acceleration

A useful technique for understanding the motion of an object in one dimension is to draw graphs of its position, velocity and acceleration against time.

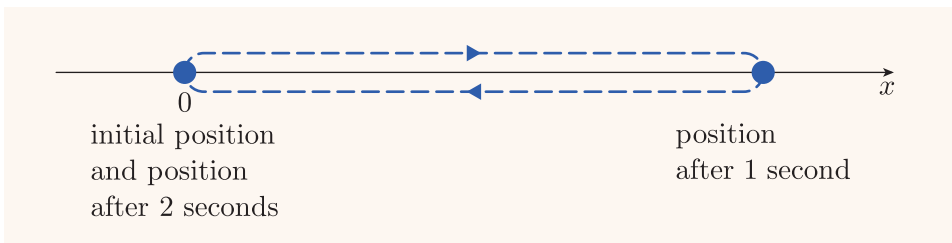
A graph of position against time is called a **position–time graph** (or sometimes a **displacement–time graph**). Similarly, a graph of velocity against time is called a **velocity–time graph**, and a graph of acceleration against time is called an **acceleration–time graph**.

For example, consider the position–time graph of a particle moving in one dimension shown in Figure 5.



**Figure 5** A position–time graph for a particle

You can see that the particle whose motion is represented by this graph starts at position  $x = 0$  at time  $t = 0$ . Its position  $x$  then increases as the time  $t$  increases until time  $t = 1$ , which means that the particle is moving in the positive  $x$ -direction during this time. Then, from time  $t = 1$  until time  $t = 2$ , the position  $x$  decreases as the time  $t$  increases, which means that the particle is moving in the negative  $x$ -direction. The particle returns to the position  $x = 0$  at time  $t = 2$ . The complete path of the particle is illustrated in Figure 6.



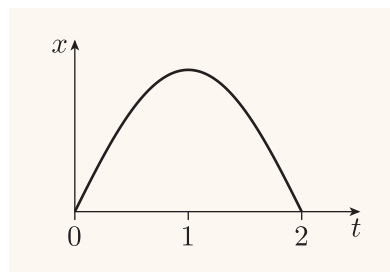
**Figure 6** A representation of the path of the object whose motion is given by the graph in Figure 5

Note that, although in general it does not make sense to describe a vector quantity as *increasing* or *decreasing*, we can describe the vector quantity position, and similarly the vector quantities velocity and acceleration, as increasing or decreasing here because for motion along a straight line these quantities are one-dimensional vectors and we are representing them by scalars. You met the ideas of increasing and decreasing position, velocity and acceleration for motion along a straight line in MST124 Units 6 and 7.

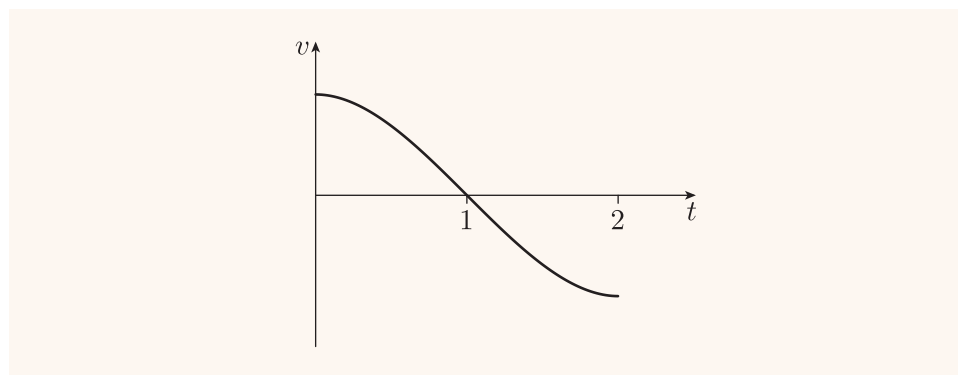
## Deducing information about velocity from a position–time graph, and about acceleration from a velocity–time graph

Since velocity is the derivative of position with respect to time, the gradient of the position–time graph of an object is the velocity of the object.

For example, consider the position–time graph in Figure 5, which is repeated in Figure 7. You can see that its gradient – that is, the velocity of the particle – is initially large and positive at  $t = 0$ , then decreases, becoming zero at  $t = 1$ , then becomes negative and continues to decrease until  $t = 2$ . The velocity–time graph for the particle is shown in Figure 8.



**Figure 7** The position–time graph for the particle

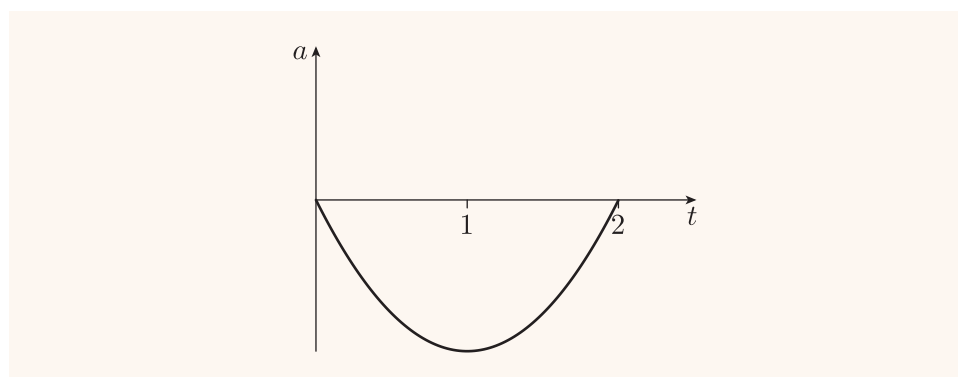


**Figure 8** The velocity–time graph for the motion depicted in Figure 5

Notice that at the instant when the particle changes from moving in the positive  $x$ -direction to moving in the negative  $x$ -direction, which is at time  $t = 1$ , the velocity is zero.

Since acceleration is the derivative of velocity with respect to time, the gradient of the velocity–time graph of an object is the acceleration of the object.

You can see that the gradient of the velocity–time graph in Figure 8 – that is, the acceleration of the particle – is initially zero and then becomes negative. It continues to decrease until time  $t = 1$ , when it starts to increase again, and eventually becomes zero again at time  $t = 2$ . The acceleration–time graph of the particle is shown in Figure 9.



**Figure 9** The acceleration–time graph for the motion depicted in Figure 5

## Deducing information about position from a velocity–time graph, and about velocity from an acceleration–time graph

Let's now look at what the velocity–time graph of an object tells you about the position of the object.

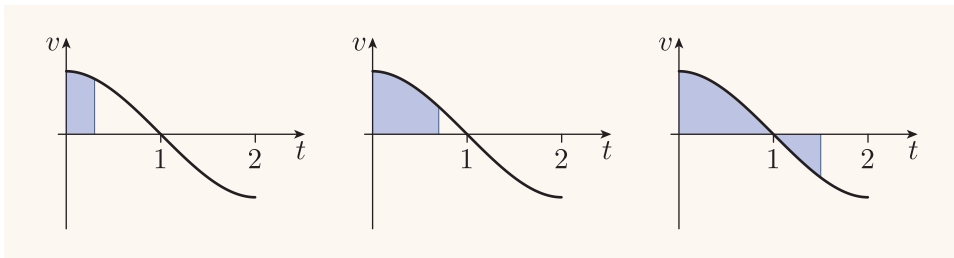
Since integrating the velocity  $v$  of an object with respect to time gives the position  $x$  of the object, the following holds for any two instants  $t_1$  and  $t_2$  in time:

$$\int_{t_1}^{t_2} v \, dt = [x]_{t_1}^{t_2} = x(t_2) - x(t_1).$$

In other words, the definite integral  $\int_{t_1}^{t_2} v \, dt$  is equal to the *change in the position* of the object from the time  $t_1$  to the time  $t_2$ . Remember from MST124 that the definite integral  $\int_{t_1}^{t_2} v \, dt$  represents the *signed area* between the velocity–time graph for the object and the time axis, from time  $t_1$  to time  $t_2$ . Provided that  $t_1 < t_2$ , areas above the time axis contribute a positive amount to the signed area, and areas below the time axis contribute a negative amount.

So a signed area between a velocity–time graph and the time axis represents a change in position.

Let's apply these facts to the velocity–time graph in Figure 8. As the time  $t$  increases, starting from  $t = 0$ , the signed area between the velocity–time graph and the time axis from time 0 to time  $t$  will change, as illustrated in Figure 10. At each time  $t$  the current signed area will be the amount that the position of the particle has changed since time  $t = 0$ .



**Figure 10** The signed area between the velocity–time graph in Figure 8 and the time axis from 0 to  $t$ , for some values of  $t$

You can see that the change in the position will increase until time  $t = 1$ , which corresponds to the particle moving in the positive  $x$ -direction. After that, there will be negative contributions to the change in position, so it will start to decrease, which corresponds to the particle moving in the negative  $x$ -direction.

Since we know (from the position–time graph in Figure 7) that the particle returns to the origin at time  $t = 2$ , the distance that it moves in the positive  $x$ -direction between time  $t = 0$  and time  $t = 1$  must be equal to the distance that it moves in the negative  $x$ -direction between time  $t = 1$  and time  $t = 2$ . So the signed area from  $t = 1$  to  $t = 2$  must be equal in magnitude but opposite in sign to the signed area from  $t = 0$  to  $t = 1$ .

Now let's look at what the acceleration–time graph of an object tells you about the velocity of the object. Since integrating the acceleration  $a$  of an object with respect to time gives the velocity  $v$  of the object, the following holds for any two instants  $t_1$  and  $t_2$  in time:

$$\int_{t_1}^{t_2} a \, dt = [v]_{t_1}^{t_2} = v(t_2) - v(t_1).$$

In other words, the definite integral  $\int_{t_1}^{t_2} a \, dt$  is equal to the *change in the velocity* of the object from the time  $t_1$  to the time  $t_2$ . The definite integral  $\int_{t_1}^{t_2} a \, dt$  represents the signed area between the acceleration–time graph and the time axis, from time  $t_1$  to time  $t_2$ .

So a signed area between an acceleration–time graph and the time axis represents a change in velocity.

For example, you can see from the acceleration–time graph in Figure 9 that, as  $t$  increases, starting from  $t = 0$ , the signed area – that is, the change in the velocity – is always negative, and decreases. This agrees with the velocity–time graph in Figure 8.

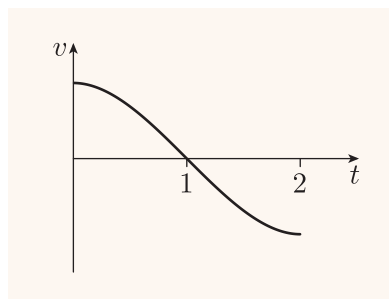
### Velocity, speed and acceleration

When you are working with graphs of position, velocity and acceleration, it is important to remember that velocity is not the same as speed. As you have seen, speed is the magnitude of velocity. For example, consider again the velocity–time graph in Figure 8, which is repeated in Figure 11. Although the velocity of the particle is decreasing throughout the whole two second time interval (except at the endpoints), the speed of the particle is decreasing between  $t = 0$  and  $t = 1$ , and increasing between  $t = 1$  and  $t = 2$ .

Note in particular that if an object has a negative acceleration, then although this means that its velocity is decreasing, it does not necessarily mean that its speed is decreasing. For example, although the particle whose motion we have been considering has a negative acceleration throughout the whole two second interval of its motion (except at the endpoints), its speed is *increasing* between  $t = 1$  and  $t = 2$ . In general, as mentioned earlier, the following facts hold.

- If the velocity and acceleration are both positive or both negative, then the speed is increasing.
- If the velocity and acceleration have opposite signs, then the speed is decreasing.

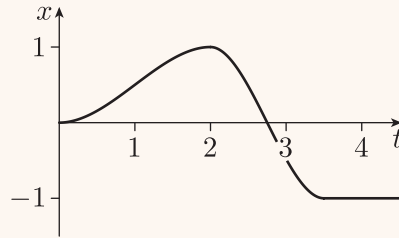
Now try the following activities to test your skills with graphs of motion.



**Figure 11** The velocity–time graph for the particle

**Activity 3** *Interpreting a position–time graph*

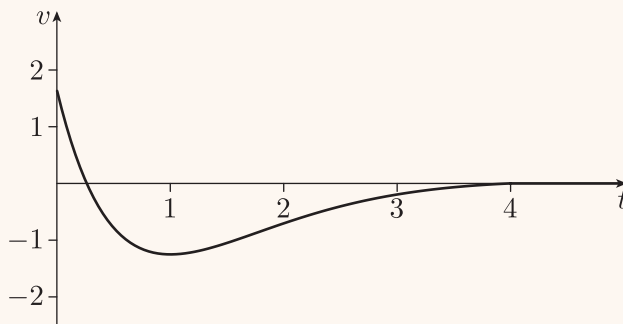
The graph below shows the position (in metres) of a particle against time (in seconds).



- At approximately what time does the direction of motion of the particle reverse?
- Approximately how far is the particle from its starting position at that time?
- At approximately what time is the speed of the particle greatest?

**Activity 4** *Interpreting a velocity–time graph*

The graph below shows the velocity (in  $\text{m s}^{-1}$ ) of a particle against time (in seconds).



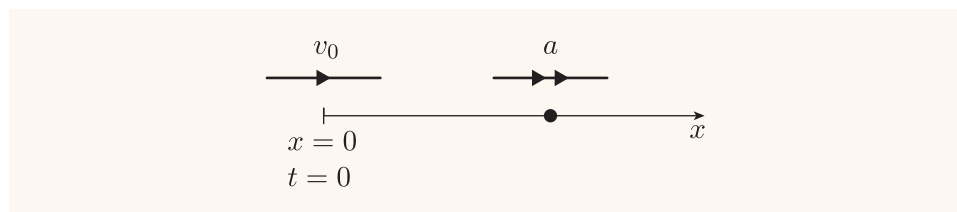
- At approximately what time is the speed of the particle greatest?
- At approximately what time does the direction of motion of the particle reverse?
- If the particle starts (at time  $t = 0$ ) at the origin, is its position positive or negative at time  $t = 4$ ?

### 1.3 Motion with constant acceleration

A situation that occurs frequently in practice is where an object moves along a straight line with *constant* acceleration. For this type of motion, we can obtain a set of fairly simple equations that relate the position, velocity and acceleration of the object, and time.

To obtain these equations, let's consider a particle moving along a straight line with constant acceleration  $a$ . We start by choosing an instant during the period of motion to measure time from; that is, we choose when time  $t = 0$  will be. It is usually convenient to choose this time to be the start of the period of motion, if possible, but any instant in the period of motion will do. Then we choose the  $x$ -axis to lie along the line of motion, with the origin at the point where the particle is at time  $t = 0$ . We can choose the positive direction of the  $x$ -axis to be in either direction along the line of motion, but if the particle moves in only one direction then it is usually convenient to choose that direction.

As usual, we denote the position and velocity of the particle by  $x$  and  $v$ , respectively, and we denote time by  $t$ . So  $x$  and  $v$  are functions of  $t$ . At time  $t = 0$ , the particle is at the origin, so its position is  $x = 0$ . We denote the velocity of the particle at time  $t = 0$  by  $v_0$ , which we refer to as the **initial velocity** of the particle. So  $v_0$  is a constant. The situation is illustrated in Figure 12.



**Figure 12** A particle moving with constant acceleration along a straight line

It is important to keep in mind that the acceleration  $a$  and the initial velocity  $v_0$  are constants, and the position  $x$  and velocity  $v$  are functions of the time  $t$ .

We can find an equation for the velocity  $v$  in terms of the time  $t$  by integrating the acceleration  $a$ , which is a constant. This gives

$$v = \int a \, dt = at + c,$$

where  $c$  is a constant. Using the initial condition that  $v = v_0$  when  $t = 0$  gives  $v_0 = 0 + c$ , so  $c = v_0$ . So we obtain the equation

$$v = v_0 + at. \tag{1}$$

We can find an equation for the position  $x$  in terms of the time  $t$  by integrating the velocity  $v$ . This gives

$$x = \int v \, dt = \int (v_0 + at) \, dt = v_0 t + \frac{1}{2}at^2 + d,$$

where  $d$  is a constant.

Using the initial condition that  $x = 0$  when  $t = 0$  gives  $0 = 0 + 0 + d$ , so  $d = 0$ . So we obtain the equation

$$x = v_0 t + \frac{1}{2} a t^2. \quad (2)$$

Notice that the velocity  $v$  is a *linear* function of time, and the position  $x$  is a *quadratic* function of time. Because the acceleration is constant, it is a constant function of time.

While equations (1) and (2) are very useful, it is sometimes convenient to use an equation that does not contain  $t$ . We can obtain such an equation by eliminating  $t$  from equations (1) and (2), as follows. Equation (1) can be rearranged as

$$t = \frac{v - v_0}{a},$$

and substituting this expression for  $t$  into equation (2) gives

$$x = v_0 \left( \frac{v - v_0}{a} \right) + \frac{1}{2} a \left( \frac{v - v_0}{a} \right)^2;$$

that is,

$$x = \frac{v_0(v - v_0)}{a} + \frac{(v - v_0)^2}{2a}.$$

Multiplying both sides by  $2a$  gives

$$\begin{aligned} 2ax &= 2v_0(v - v_0) + (v - v_0)^2 \\ &= 2v_0v - 2v_0^2 + v^2 - 2v_0v + v_0^2 \\ &= v^2 - v_0^2. \end{aligned}$$

The equation obtained here is usually written as

$$v^2 = v_0^2 + 2ax.$$

Two further useful equations can be obtained by eliminating  $a$  and  $v_0$  in turn from equations (1) and (2). You are asked to do this in the next activity.

### Activity 5 Deriving equations of motion for constant acceleration along a straight line

By eliminating  $a$  and  $v_0$  in turn from equations (1) and (2), derive the following formulas for the position  $x$  of a particle that moves with constant acceleration along a straight line, in terms of its velocity  $v$ , its acceleration  $a$  and time  $t$ , where  $x = 0$  when  $t = 0$ , and  $v = v_0$  when  $t = 0$ .

(a)  $x = \frac{1}{2}(v_0 + v)t$       (b)  $x = vt - \frac{1}{2}at^2$

Notice that  $\frac{1}{2}(v_0 + v)$  is simply the average of the initial velocity and the current velocity, so, for motion with constant acceleration along a straight line, the position is simply the average velocity multiplied by the time taken.

The five equations of motion that you have met in this subsection are summarised below.

### Equations of motion for constant acceleration along a straight line

If a particle is moving along a straight line with constant acceleration  $a$ , with position  $x$  and velocity  $v$  at time  $t$ , and with  $x = 0$  and  $v = v_0$  at time  $t = 0$ , then the following equations hold.

$$\begin{array}{lll} x = v_0 t + \frac{1}{2} a t^2 & x = \frac{1}{2} (v_0 + v) t & x = v t - \frac{1}{2} a t^2 \\ v = v_0 + a t & v^2 = v_0^2 + 2 a x & \end{array}$$

It is worth reflecting for a moment on what these equations mean. Remember that  $x$  and  $v$  are functions of time  $t$ . So, for example, the equation  $v^2 = v_0^2 + 2ax$  can be written as  $v(t)^2 = v_0^2 + 2ax(t)$ . This emphasises the fact that the symbols  $x$  and  $v$  in the equations mean the position and velocity, respectively, *evaluated at the same time  $t$* . Whenever you apply the equations to a particular example of motion with constant acceleration, you will always use the same value of  $a$ , and the same value of  $v_0$ , no matter what point in the motion you need to consider. However, the values of  $x$ ,  $v$  and/or  $t$  will differ depending on what point in the motion you are considering. Sometimes you may need to consider two or more different points in the motion, so you will need to use the values of  $t$ ,  $x$  and/or  $v$  that apply to the first point, then the different values of  $t$ ,  $x$  and/or  $v$  that apply to the second point, and so on.

Another thing to remember about the equations is that they hold only if the particle has position  $x = 0$  at time  $t = 0$ . This is because we made this assumption in deriving the equations. You can always position the  $x$ -axis to ensure that this condition holds.

Notice that each of the five equations omits one of the five quantities  $a$ ,  $v_0$ ,  $t$ ,  $x$  and  $v$ . When you are solving a problem involving constant acceleration along a straight line, you usually have values for three of these five quantities. By choosing the appropriate equation you can calculate the value of either of the other two quantities.

The next example demonstrates how to use the equations to solve a problem involving constant acceleration.



**Example 2** Using the equations of motion for constant acceleration

A car accelerates from rest with a constant acceleration of  $3 \text{ m s}^{-2}$  along a straight road.

- What is the velocity of the car after 5 seconds?
- How far does the car travel in this time?

**Solution**

State any modelling assumptions.

Model the car as a particle.

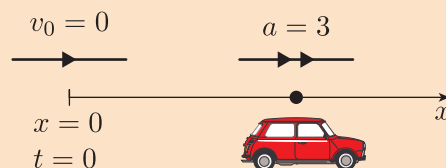
The acceleration is constant, so aim to use the equations of motion for constant acceleration. Choose when to measure time from, and choose the position of the  $x$ -axis, making sure that the origin is at the point where the car is at time  $t = 0$ .

Measure time from the moment the car begins accelerating.

Choose the positive direction of the  $x$ -axis to be the direction in which the car accelerates, with the origin at the starting point of the car.

Draw a diagram, including the  $x$ -axis and relevant information.

The position of the car at time  $t$  is  $x$ . Since the car starts from rest, its initial velocity is  $v_0 = 0$ . Its acceleration is  $a = 3$ .



- Identify the three quantities for which you have values, and the quantity whose value you want to find. Hence choose an appropriate equation of motion to use.

Here we take  $t = 5$ . So we know values for  $v_0$ ,  $a$  and  $t$ , and we want to find the corresponding value of the velocity  $v$ .

The velocity  $v$  at time  $t$  is given by  $v = v_0 + at$ , so at  $t = 5$  we have

$$v = 0 + 3 \times 5 = 15.$$

The velocity of the car after 5 seconds is  $15 \text{ m s}^{-1}$ .



City traffic is nearly always speeding up or slowing down

(b)  Proceed as in part (a). 

Again we take  $t = 5$ . We know values for  $v_0$ ,  $a$  and  $t$ , and we want to find the corresponding value of the position  $x$ .

The position  $x$  at time  $t$  is given by  $x = v_0 t + \frac{1}{2} a t^2$ , so at  $t = 5$  we have

$$x = 0 \times 5 + \frac{1}{2} \times 3 \times 5^2 = 37.5.$$

The car travels 37.5 metres during the 5 seconds.

Here is an activity involving constant acceleration for you to try. To solve the problem, remember to identify the quantities that you know and the quantity that you want to find, and hence choose an appropriate equation of motion to use.

### Activity 6 *Using an equation of motion for constant acceleration*

A train accelerates from rest with a constant acceleration of  $0.3 \text{ m s}^{-2}$ . What is the velocity of the train after it has travelled 35 m? Give your answer in  $\text{m s}^{-1}$  to two significant figures.

In Example 2 and Activity 6, we started by knowing information relating to the beginning of the period of constant acceleration (namely the position and the velocity of the moving object at this time), and we had to find information relating to a later time. In the next example, we start by knowing information relating to a later time.



Only time for a single run!

### Example 3 *Using the equations of motion for constant acceleration*

A cricketer hits a cricket ball along the ground. The ball rolls in a straight line, slowing down at the rate of  $2 \text{ m s}^{-2}$ , and comes to a halt after 10 seconds. Find the following.

- The speed with which the ball left the bat.
- The distance travelled by the ball.

#### Solution

 State any modelling assumptions. 

Model the ball as a particle.

 The acceleration is constant, so aim to use the equations of motion for constant acceleration. Choose when to measure time from, and choose the position of the  $x$ -axis, making sure that the origin is at the point where the ball is at time  $t = 0$ . 

Measure time from the moment the bat hits the ball.

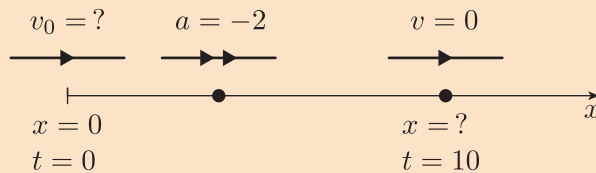
Choose the positive direction of the  $x$ -axis to be the direction in which the ball moves, with the origin at the point where the ball leaves the bat.

 Draw a diagram, including the  $x$ -axis and relevant information. 

We know that the velocity  $v$  of the ball is zero at time 10 seconds after it is hit, so  $v = 0$  when  $t = 10$ .

 Since the velocity of the ball is positive (because the ball moves in the positive direction of the  $x$ -axis), and the ball slows down, its acceleration is negative. 

The constant acceleration of the ball is  $a = -2$ .



- (a)  Identify the three quantities for which you have values, and the quantity whose value you want to find. Hence choose an appropriate equation of motion to use. 

We know that  $a = -2$ , and  $v = 0$  when  $t = 10$ . We want to find the initial velocity  $v_0$ .

Using the equation  $v = v_0 + at$  gives

$$0 = v_0 + (-2) \times 10,$$

so  $v_0 = 20$ . Hence the ball leaves the bat with speed  $20 \text{ m s}^{-1}$ .

- (b)  Proceed as in part (a). 

Again, we know that  $a = -2$ , and  $v = 0$  when  $t = 10$ . We want to find the position  $x$  of the ball at the same time  $t = 10$ .

Using the equation  $x = vt - \frac{1}{2}at^2$  gives

$$x = 0 \times 10 - \frac{1}{2} \times (-2) \times 10^2 = 10^2 = 100.$$

Hence the ball travels 100 m.

Notice that, since we determined the value of  $v_0$  in part (a) of Example 3, we could have used an equation of motion containing  $v_0$  to determine the value of  $x$  in part (b). However it is usually best to work with known information, rather than calculated information, where possible. One reason is that the calculated information may contain errors. Another reason is that if the calculated information was calculated only approximately, it could lead to rounding errors in the later calculations.

Here is a similar problem for you to try.

### Activity 7 *Using the equations of motion for constant acceleration*



Runways take up a lot of space, so aircraft manoeuvres are carefully planned to fit in

An aircraft touches down with a speed of  $70 \text{ m s}^{-1}$  at the start of a runway of length 550 metres. To ensure safe movement from the runway to the terminal building, the pilot is required to reduce the speed of the aircraft to  $5 \text{ m s}^{-1}$  at the end of the runway.

- Calculate the magnitude of the constant acceleration (the slowing down) needed during braking to reduce the speed as required.
- How long does it take the aircraft to travel the length of the runway?

Give your answers to two significant figures.

## 1.4 Vertical motion under gravity

You learned in Unit 5 that in the absence of forces such as air resistance, every object dropped from reasonably close to the surface of the Earth falls under the influence of gravity with the same constant acceleration. This acceleration is known as the **acceleration due to gravity**. It is directed vertically downwards, and its magnitude, which we denote by  $g$ , is  $9.8 \text{ m s}^{-2}$  to two significant figures. We will use this approximation for  $g$  throughout this unit.

Since the acceleration due to gravity is constant, you can use the equations of motion for constant acceleration along a straight line to solve problems involving objects moving vertically under the influence of gravity.

You can try this in the next activity. Since the motion is vertically downwards, take the  $x$ -axis to point vertically downwards.

### Activity 8 *Solving a problem about the motion of a dropped object*

A small stone is dropped down an empty well and falls under the influence of gravity alone. The bottom of the well is 30 m below the point from which the stone is released. Find the following, to two significant figures.

- How long it takes for the stone to reach the bottom of the well.
- The speed of the stone as it hits the bottom of the well.

In Activity 8, the stone was falling vertically downwards. The next example is about a ball that is thrown vertically upwards. As you know from experience, the ball will first go up, then start falling down. In both the upwards and downwards parts of its motion, it is moving under the influence of gravity, so its acceleration is the acceleration due to gravity.

Because the acceleration is constant throughout the motion, the equations of motion for constant acceleration along a straight line apply throughout the motion, with the same value of the acceleration  $a$  and the same value of the initial velocity  $v_0$ , and with the values of the position  $x$  and the velocity  $v$  changing as the time  $t$  increases, in the usual way. There is no need to consider the upwards and downwards parts of the motion separately.

In solving the problem in the example, we take the  $x$ -axis to lie along the line of motion, as usual. However, because the ball moves both upwards and downwards during its motion, each of the two possible choices for the positive direction of the  $x$ -axis is just as convenient as the other. In the example we will take the  $x$ -axis to point upwards. This means that we have to take the acceleration due to gravity, which points downwards when considered as a vector, to be *negative*.

#### Example 4 *Solving a problem about upwards then downwards motion under gravity*

A man leaning over the edge of a balcony throws a ball vertically upwards from a point 5 m above the ground. The initial speed of the ball is  $8 \text{ m s}^{-1}$ . The ball travels up and then down in a straight vertical line under the influence of gravity alone, and eventually hits the ground. Find the following.

- (a) The maximum height reached by the ball (measured from the ground).
- (b) The time that elapses before the ball hits the ground.

Give your answers to two significant figures.

#### Solution

 State any modelling assumptions. Choose when to measure time from. Draw a diagram, choose the position of the  $x$ -axis (ensuring that  $x = 0$  when  $t = 0$ ), and include relevant information. 

Model the ball as a particle.

Measure time from the moment the ball is released.

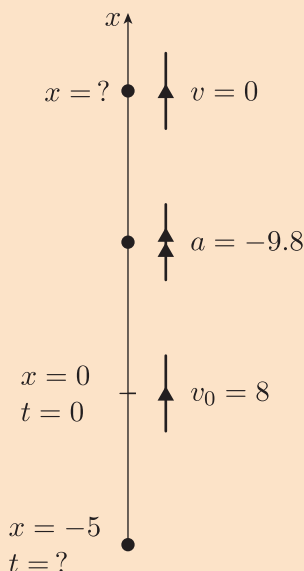
Choose the positive direction of the  $x$ -axis to be upwards, with the origin at the point from where the ball is released, which is 5 m above the ground.

We know that the initial velocity of the ball is  $v_0 = 8$ , and that its acceleration is the acceleration due to gravity, which has magnitude  $9.8 \text{ m s}^{-2}$  and is directed vertically downwards.



Since the acceleration is in the opposite direction to the positive direction of the  $x$ -axis, it is negative.

So the acceleration is  $a = -9.8$ .



- (a) The maximum height reached by the ball occurs when it stops moving upwards and starts moving downwards. This happens when its velocity changes from being positive to being negative.

The ball reaches its maximum height when its velocity  $v$  is zero.

Choose and apply an appropriate equation of motion.

Here we know that  $v_0 = 8$  and  $a = -9.8$ , and we want to find the value of the position  $x$  when  $v = 0$ .

Using  $v^2 = v_0^2 + 2ax$  gives

$$0 = 8^2 + 2(-9.8)x,$$

so

$$x = \frac{8^2}{2 \times 9.8} = 3.26 \dots$$

So the maximum height reached by the ball is 3.3 m (to 2 s.f.) above the point of release. Since the point of release is 5 m above the ground, the maximum height reached by the ball is  $3.3 + 5 = 8.3$  m (to 2 s.f.).

- (b) The ball hits the ground when  $x = -5$ .

 Choose and apply an appropriate equation of motion. 

Here we know that  $v_0 = 8$  and  $a = -9.8$ , and we want to find the value of the time  $t$  when  $x = -5$ .

Using  $x = v_0t + \frac{1}{2}at^2$  gives

$$-5 = 8t + \frac{1}{2}(-9.8)t^2;$$

that is,

$$4.9t^2 - 8t - 5 = 0.$$

 Solve this quadratic equation by using the quadratic formula. 

This gives

$$\begin{aligned} t &= \frac{8 \pm \sqrt{8^2 - 4 \times 4.9 \times (-5)}}{2 \times 4.9} \\ &= \frac{8 \pm \sqrt{162}}{9.8}. \end{aligned}$$

Since the time taken for the ball to hit the ground must be positive, and  $\sqrt{162} > 8$ , the required value of  $t$  is

$$t = \frac{8 + \sqrt{162}}{9.8} = 2.11 \dots$$

So the time taken for the ball to hit the ground is 2.1 s (to 2 s.f.).

Here is a similar problem for you to try.

### Activity 9 Solving a problem about upwards then downwards motion under gravity

A ball is thrown vertically upwards from the ground with speed  $25 \text{ m s}^{-1}$  and moves under the influence of gravity alone.

- (a) Find the following, to two significant figures.
- The maximum height above the ground reached by the ball.
  - The time taken for the ball to reach its maximum height.
  - The time taken for the ball to return to the ground.
- (b) Write down formulas for the velocity and position of the ball in terms of the time  $t$ , and sketch graphs of the acceleration, velocity and position of the ball against time.

In Activity 9, the time that the ball takes to reach the ground is exactly twice the time that it takes to reach its maximum height. This is because the motion is symmetrical. The position is a quadratic function of time, so the position–time graph of the motion (shown in the solution to the activity) is a parabola with a vertical axis of symmetry.

## 1.5 Piecewise constant acceleration

In this subsection we will consider examples of motion along a straight line in which the acceleration has a constant value for a period of time, then switches to a different constant value for the next period of time, then perhaps switches again for another period of time, and so on. So the motion consists of several stages, each with a possibly different constant acceleration. We say that the moving object moves with **piecewise constant acceleration**. The position and velocity of the object at the start of each new stage of the motion are equal to its position and velocity at the end of the previous stage.

You can solve problems involving piecewise constant acceleration by applying the equations of motion for constant acceleration to each stage of the motion separately. However, you have to be careful, because the equations apply only when the position is  $x = 0$  at time  $t = 0$ . So when you apply an equation of motion to a stage of the motion, you have to temporarily take the time  $t$  and the position  $x$  to be the time and the position referred to the start of that stage of the motion, and then interpret what your results tell you about the whole motion.

In such problems, it is often helpful to sketch a position–time graph and/or a velocity–time graph of the motion, to help you refer easily to the information that you know. This is done in the following example.



A British metro tram

### Example 5 Working with piecewise constant acceleration

Two tram stops are 1 km apart and linked by a straight tramline. A tram leaves the first tram stop with an acceleration of  $1.25 \text{ m s}^{-2}$  and travels with this constant acceleration until it reaches its top speed of  $20 \text{ m s}^{-1}$ . It then travels at this speed until it needs to slow down to stop at the second tram stop. The slowing down happens at a constant rate of  $1 \text{ m s}^{-2}$ .

Find the time taken for the tram to travel from the first tram stop to the second one.



**Solution**

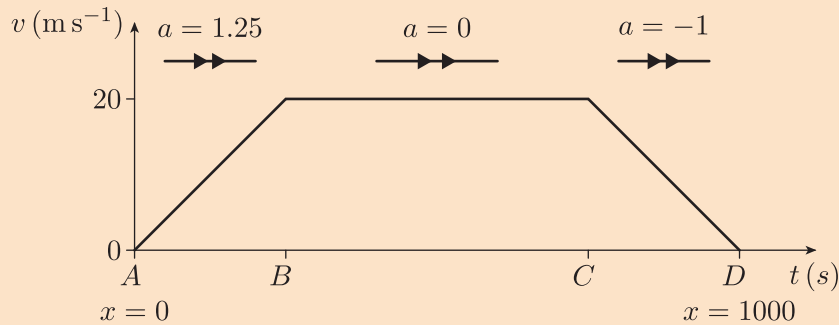
State any modelling assumptions, draw a diagram and include relevant information. In this example, it is convenient to summarise the whole problem on a velocity–time graph.

Model the tram as a particle.

Let  $A$  and  $D$  be the first and second tram stops, and let  $B$  and  $C$  be the points in the tram's journey where its acceleration changes.

A velocity–time graph for the journey is shown below.

The known accelerations, and the maximum velocity of the tram, are shown. We know that the total distance travelled is 1000 m.

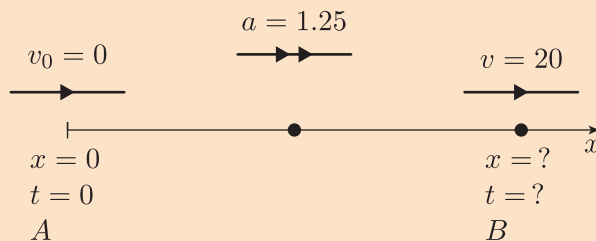


Consider the three stages of the motion separately. Deal with the stages  $AB$  and  $CD$  first, in each case finding the distance travelled as well as the time taken. This is because, to work out the time taken in the stage  $BC$ , we need the distance travelled in that stage, which we can find by subtracting the distances travelled in the stages  $AB$  and  $CD$  from the total distance between the tram stops.

**Stage  $AB$** 

Choose when to measure time from, choose the position of the  $x$ -axis and draw a diagram.

For this stage, measure time from when the tram is at point  $A$ , and take the  $x$ -axis to point in the direction of motion, with the origin at  $A$ .



Choose and apply appropriate equations of motion.

We know that the initial velocity is  $v_0 = 0$  and the acceleration is  $a = 1.25$ , and we want to find the time  $t$  and the position  $x$  when the velocity is  $v = 20$ .

Using  $v = v_0 + at$  gives

$$20 = 0 + 1.25t,$$

so

$$t = \frac{20}{1.25} = 16.$$

So the time taken for stage  $AB$  is 16 s.

Using  $v^2 = v_0^2 + 2ax$  gives

$$20^2 = 0^2 + 2 \times 1.25x,$$

so

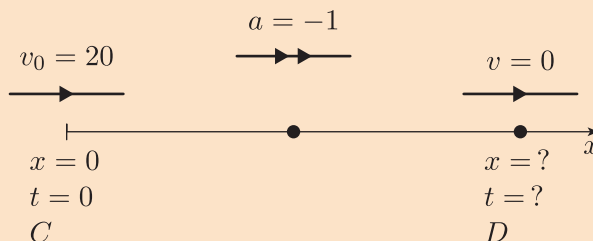
$$x = \frac{20^2}{2 \times 1.25} = 160.$$

So the distance travelled during stage  $AB$  is 160 m.

### Stage $CD$

Choose when to measure time from, choose the position of the  $x$ -axis and draw a diagram.

For this stage, measure time from when the tram is at point  $C$ , and take the  $x$ -axis to point in the direction of motion, with the origin at  $C$ .



Choose and apply appropriate equations of motion.

We know that the initial velocity is  $v_0 = 20$  and the acceleration is  $a = -1$ , and we want to find the time  $t$  and the position  $x$  when the velocity is  $v = 0$ .

Using  $v = v_0 + at$  gives

$$0 = 20 + (-1)t,$$

so  $t = 20$ . So the time taken for stage  $CD$  is 20 s.

Using  $v^2 = v_0^2 + 2ax$  gives

$$0^2 = 20^2 + 2(-1)x,$$

so

$$x = \frac{20^2}{2} = 200.$$

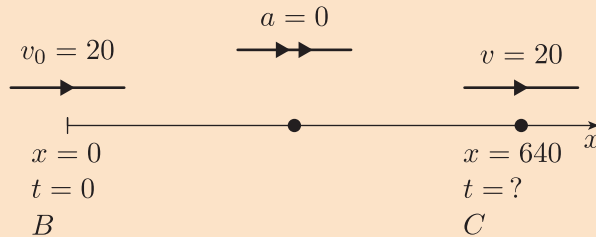
So the distance travelled during stage  $CD$  is 200 m.

### Stage $BC$

 Choose when to measure time from, choose the position of the  $x$ -axis and draw a diagram. 

For this stage, measure time from when the tram is at point  $B$ , and take the  $x$ -axis to point in the direction of motion, with the origin at  $B$ .

The distance travelled during this stage is  $1000 - 160 - 200 = 640$  m.



 Choose and apply appropriate equations of motion. 

We know that the initial velocity is  $v_0 = 20$  and the acceleration is  $a = 0$ , and we want to find the time  $t$  when the position is  $x = 640$ .

Using  $x = v_0t + \frac{1}{2}at^2$  gives

$$640 = 20 \times t + 0,$$

so

$$t = \frac{640}{20} = 32.$$

So the time taken for stage  $BC$  is 32 s.

 Now complete the problem. 

### Complete journey

The total time taken by the tram is the sum of the times taken for the three stages, which is (in seconds)

$$16 + 32 + 20 = 68.$$

Hence the tram takes 68 s to travel from the first tram stop to the second one.



When you play with a yo-yo you find yourself applying a sudden upwards acceleration as it reaches its lowest position

Notice that for stage *BC* in Example 5 above we applied an equation of motion for constant acceleration to motion with an acceleration of zero; that is, to motion with constant speed. This is fine, since an acceleration of zero is a constant acceleration. However, an alternative way to deal with such an instance is just to use the usual relationships that apply to motion with constant speed. Here, since the tram travels 640 m at a constant speed of  $20 \text{ m s}^{-1}$ , the time that it takes is  $640/20$  seconds; that is, 32 seconds, as found in Example 5.

Here is an activity similar to Example 5 for you to try.

### Activity 10 Working with piecewise constant acceleration.

A child spins a yo-yo vertically down to its lowest position. The motion has two stages, each with constant acceleration. In the first stage, the yo-yo speeds up. In the second stage, it slows down until its speed is (momentarily) zero at its lowest position.

The yo-yo starts from rest, and reaches its maximum velocity of  $0.4 \text{ m s}^{-1}$  downwards after 5 seconds. Its velocity reaches zero again after a further 0.3 seconds.

Find, to two significant figures, the distance between the highest and lowest positions of the yo-yo, and the maximum magnitude of the acceleration reached by the yo-yo.

## 1.6 Velocity and acceleration as functions of position

In the previous three subsections we considered motion along a straight line with constant acceleration. In this subsection we return to considering motion along a straight line where the acceleration can change.

Earlier you saw that the position, velocity and acceleration of an object moving along a straight line are related by the differential equations

$$v = \frac{dx}{dt} \quad \text{and} \quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

These differential equations have time as the independent variable. There is an alternative differential equation, relating velocity and acceleration, which has position as the independent variable. It can be obtained as follows.

Consider a particle moving along a straight line, with position  $x$  at time  $t$ . Suppose that instead of knowing the velocity as a function of time, as  $v = v(t)$ , we know the velocity as a function of *position*, as  $v = v(x)$ . That is, we know the velocity of the particle when it is at any particular position along the line. Since  $v$  is a function of  $x$ , and  $x$  is a function of  $t$ , the chain rule gives

$$\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt}.$$

The derivative  $dv/dt$  on the left-hand side of this equation is the acceleration  $a$ , and the derivative  $dx/dt$  that appears second on the right-hand side is the velocity  $v$ . Using these facts, and swapping the order on the right-hand side, gives the equation below.

### Alternative relationship between velocity and acceleration

If a particle moves along a straight line with velocity  $v$  and acceleration  $a$  at position  $x$ , then

$$a = v \frac{dv}{dx}.$$

If you know the velocity of an object as a function of position, then you can use this equation to obtain its acceleration as a function of position. You can also use the equation the other way round, to obtain the velocity of the object as a function of position when you know its acceleration as a function of position, though this involves solving a differential equation. The process is illustrated in the next example.

This example is about the motion of a snowboarder after he hits a patch of wet snow, which slows him down. You saw a model for such motion in Example 1 in Subsection 1.1, which involved velocity and acceleration as functions of time. The example below involves a different model, in which the acceleration is given as a function of position. This models a situation in which the acceleration depends on the condition of the snow, which varies with position.



### Example 6 *Determining velocity from acceleration as a function of position*

A snowboarder slides down a straight slope, into a patch of wet snow which slows him down. The condition of the wet snow becomes worse with distance down the slope, and so slows the snowboarder down increasingly quickly. Eventually he comes to a halt.

The snowboarder's velocity at the time when he hits the start of the wet snow is  $25 \text{ m s}^{-1}$ . While he is still moving, he slows down at the rate of  $x/36$  (in metres per second per second), where  $x$  is the distance (in metres) down the slope measured from the start of the wet snow.

Find the following.

- The velocity of the snowboarder as a function of position, during the period of slowing down.
- The distance of the snowboarder from the start of the patch of wet snow at the time when he comes to a halt.

### Solution

State any modelling assumptions. Draw a diagram. In this problem, the direction of the  $x$ -axis and the location of the origin are suggested in the question.

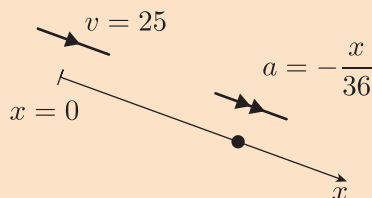
Model the snowboarder as a particle.

The  $x$ -axis points down the slope, with the origin at the start of the wet snow.

We know that the velocity is  $v = 25$  when the position is  $x = 0$ .

During the period of slowing down, the snowboarder is moving in the positive  $x$ -direction and slowing down, so his acceleration will be negative at all positions involved. Since the expression  $x/36$  for the magnitude of the acceleration is positive for all values of  $x$  involved, the acceleration must be equal to the negative of this expression.

The acceleration is given by  $a = -x/36$ .



- (a) We know the acceleration as a function of position and we want to determine the velocity as a function of position, so we use the equation  $a = v(\mathrm{d}v/\mathrm{d}x)$ .

Using the equation  $a = v \frac{\mathrm{d}v}{\mathrm{d}x}$  gives

$$-\frac{x}{36} = v \frac{\mathrm{d}v}{\mathrm{d}x};$$

that is,

$$v \frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{x}{36}.$$

To determine  $v$  as a function of  $x$  we have to solve this differential equation. It is separable, so it can be solved using separation of variables.

Separating the variables gives

$$\int v \, \mathrm{d}v = - \int \frac{x}{36} \, \mathrm{d}x.$$

Performing the integrations gives

$$\frac{v^2}{2} = -\frac{x^2}{72} + c,$$

where  $c$  is a constant.

Use the known initial condition to find the value of  $c$ .

Since  $v = 25$  when  $x = 0$ , we have

$$\frac{25^2}{2} = -\frac{0}{72} + c$$

so

$$c = \frac{25^2}{2} = \frac{625}{2}.$$

Hence the velocity is given by

$$\frac{v^2}{2} = -\frac{x^2}{72} + \frac{625}{2};$$

that is,

$$v^2 = -\frac{x^2}{36} + 625.$$

Since the snowboarder is moving in the positive  $x$ -direction, his velocity will be positive for all values of  $x$  involved. Hence the velocity is given by

$$v = \sqrt{625 - \frac{x^2}{36}}.$$

- (b)  The snowboarder comes to a halt when his velocity is zero. 

The snowboarder's velocity is zero when

$$625 - \frac{x^2}{36} = 0,$$

which gives

$$x^2 = 625 \times 36 = 22\,500.$$

Since the snowboarder's position is positive when he comes to a halt,

$$x = \sqrt{22\,500} = 150.$$

The snowboarder comes to a halt 150 m into the patch of wet snow.

The next activity is about magnetic attraction. A magnet causes attracted objects to accelerate towards it. The closer an attracted object is to the magnet, the greater is the acceleration caused by the magnet.

Note that in this activity it is convenient to take the  $x$ -axis to point in the *opposite* direction to the direction of motion, as specified in the activity.

### Activity 11 *Determining velocity from acceleration as a function of position*

A tiny ball bearing is attracted to a spherical magnet of radius 1 cm. When the ball bearing is distance  $x$  (in cm) from the centre of the magnet, the acceleration of the ball bearing has magnitude

$$\frac{24}{x^2}$$

(in  $\text{cm s}^{-2}$ ) and is directed towards the centre of the magnet.

The ball bearing is held at rest 4 cm from the centre of the magnet, and then released. Find its speed when it hits the magnet.

In your working, take the  $x$ -axis to point in the direction from the centre of the magnet to the ball bearing, with the origin at the centre of the magnet. This accords with the use of the variable  $x$  above.



## 2 Newton's second law of motion

In Section 1 you learned about the kinematics of objects moving along a straight line; that is, you learned about the relationships between the position, velocity and acceleration of objects with such motion. In this section we look at what *causes* objects to accelerate.

In Unit 5 you met Newton's first law of motion, which is as follows:

A particle remains at rest or continues to move with a constant velocity, unless acted on by a resultant force.

If a particle *is* acted on by a resultant force, then its velocity will change; that is, it will accelerate.

In this section, you will see how you can use information about the forces that act on an object to determine the acceleration of the object. You can do this using *Newton's second law of motion*, which you will see stated shortly. Once you know the acceleration of the object, you can use the methods that you learned in Section 1 to predict how the object moves.

The brilliance of Isaac Newton's mathematics was not matched by his lecturing style, according to Humphrey Newton, who served as Newton's amanuensis in Cambridge during the 1680s. In a letter to John Conduitt, dated 17 January 1727/8, he stated:

So few went to hear him, and fewer understood him, that oftimes he did, for want of hearers, read to the walls. He usually stayed about half an hour; when he had no auditors he commonly returned in a quarter of that time.

As in Section 1, throughout this section we will only consider problems in which the motion occurs along a straight line. We start in Subsection 2.1 by considering problems in which we already know the resultant force acting on an object. Then in Subsections 2.2 and 2.3 we will look at problems in which we first need to determine the resultant force, using methods similar to those that you used in Unit 5.

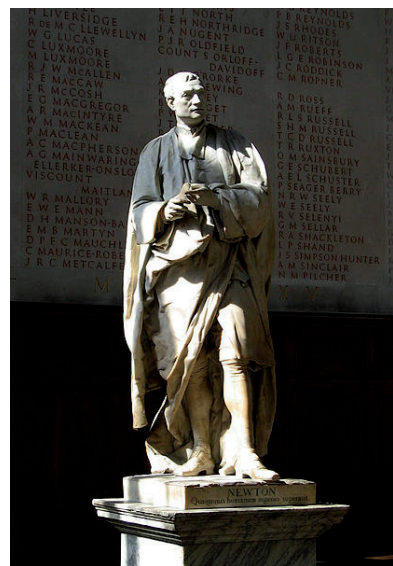
### 2.1 Effect of a non-zero resultant force

In Unit 5 you saw that a force can be defined as follows.

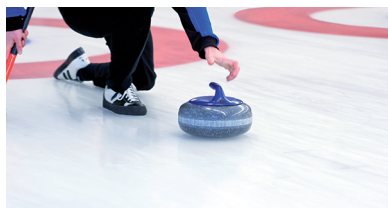
A **force** is an influence that can cause an object to accelerate. It is a vector quantity.

Intuitively, you can think of a force as a push or a pull.

Remember that the SI unit for force is the newton (N). One newton is the size of the force that causes an object of mass 1 kg to accelerate by  $1 \text{ m s}^{-2}$  in the direction of the force.



A statue of Isaac Newton in the chapel of Trinity College, Cambridge



A curling stone sliding on ice

To help you get a feel for how forces cause objects to accelerate, imagine a curling stone – a smooth stone used in the sport of curling – lying on the icy horizontal surface of a curling rink. We will assume that friction between the stone and the icy surface is negligible, and ignore it.

First suppose that the stone is at rest. Then the forces acting on it, namely its weight and the normal reaction from the surface, which both act vertically, balance each other.

Now imagine pushing the stone horizontally. No force balances your push, so the stone will start to accelerate in the direction in which you are pushing. From your experience, you know that the harder you push the stone, the greater will be its acceleration, and you also know that for a particular pushing force, the heavier the stone is, the smaller will be its acceleration. So the direction of the acceleration of the stone is the same as the direction of the force that you are applying, and the magnitude of the acceleration depends on both the force that you are applying and the mass of the stone.

Remember that the **resultant force** (also called the **total force** or **net force**) acting on an object is the vector sum of all the individual forces that act on the object. When you are pushing the curling stone, it is acted on by three forces, namely its weight, the normal reaction from the surface and the pushing force (we are ignoring friction), but the resultant force is equal to the pushing force because the other two forces balance each other.

In general, the relationship between the resultant force that acts on an object and the acceleration of the object is given by *Newton's second law of motion*, as follows.

### Newton's second law of motion

If a particle of constant mass  $m$  is acted on by a resultant force  $\mathbf{F}$ , then its acceleration  $\mathbf{a}$  is given by

$$\mathbf{F} = m\mathbf{a}.$$

It is important to appreciate that both the resultant force  $\mathbf{F}$  and the acceleration  $\mathbf{a}$  in Newton's second law are *vectors*. Like the position, velocity and acceleration of an object, the resultant force  $\mathbf{F}$  acting on the object can change with time, so it is a *vector function* of time.

The equation in Newton's second law can be rearranged as

$$\mathbf{a} = \frac{1}{m}\mathbf{F}.$$

So the law tells you that if a particle is acted on by a resultant force  $\mathbf{F}$ , then the acceleration  $\mathbf{a}$  of the object has the same direction as  $\mathbf{F}$ , and magnitude given by

$$|\mathbf{a}| = \frac{|\mathbf{F}|}{m}.$$

This accords with the curling stone example above: you can see that if the magnitude of the force  $\mathbf{F}$  is increased, then the magnitude of the acceleration  $\mathbf{a}$  also increases, and if the mass  $m$  of the object is increased, then the magnitude of the acceleration  $\mathbf{a}$  decreases.

Note that when you stop pushing the curling stone, the forces on it balance again, and hence by Newton's second law it stops accelerating. It continues to move with a constant velocity, as governed by Newton's first law. (In reality, there will be a small unbalanced force on the stone due to friction, which will slow it down at a small rate.)

As mentioned earlier, in this section we will only consider examples in which an object moves along a straight line. In such a case, even though the resultant force that acts on the object is a vector, we can represent it by a scalar, with direction indicated by a plus or minus sign, in the same way as we do for position, velocity and acceleration.

To do this, as usual we take the  $x$ -axis to lie along the line of motion. Then the resultant force  $\mathbf{F}$  acting on the object is directed along the  $x$ -axis, so it is a vector of the form  $\mathbf{F} = F\mathbf{i}$ , where  $\mathbf{i}$  is the Cartesian unit vector in the positive direction of the  $x$ -axis, and  $F$  is the  $\mathbf{i}$ -component of  $\mathbf{F}$ . We represent the force  $\mathbf{F}$  simply by the scalar  $F$ , which can be positive, negative or zero. We refer to this scalar  $F$  as a force, even though a force is a vector. The force  $F$  is a scalar function of time.

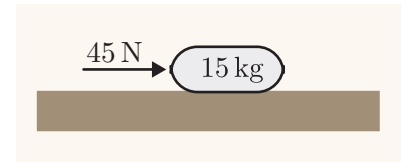
When we represent one-dimensional force vectors and acceleration vectors by scalars in this way, the equation in Newton's second law becomes

$$F = ma,$$

where  $F$  is the resultant force acting on the object,  $m$  is its mass and  $a$  is its acceleration.

For example, suppose that a curling stone of mass 15 kg is being pushed along a straight line by a resultant horizontal force of magnitude 45 N, as illustrated in Figure 13. If we take the  $x$ -axis to point in the direction of the motion, then the resultant force acting on the stone is 45 N, and hence, by Newton's second law, the acceleration of the stone is  $45/15 = 3 \text{ m s}^{-2}$ . So the acceleration is in the same direction as the force (since both are positive), and has magnitude  $3 \text{ m s}^{-2}$ .

Here are some examples and activities that illustrate how you can apply Newton's second law to solve problems involving motion along a straight line. In the first example, you will see yet another possible model for the motion of a snowboarder! Unlike with the previous two models, the snowboarder in this example is speeding up from rest, rather than slowing down due to a patch of wet snow.



**Figure 13** A curling stone pushed by a horizontal force



### Example 7 Determining force from position

A snowboarder of mass 70 kg accelerates from rest down a straight slope. Her distance  $x$  (in metres) down the slope from her starting point is given in terms of the time  $t$  (in seconds) since she left her starting point by

$$x = 25(t - 1 + e^{-t}).$$

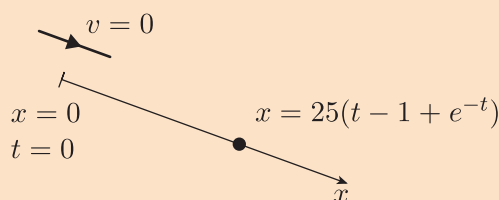
Find the resultant force acting on the snowboarder in terms of the time  $t$ .

#### Solution

State any modelling assumptions. Draw a diagram, taking the position of the  $x$ -axis to be as implied by the question, and include relevant information.

Model the snowboarder as a particle.

The positive direction of the  $x$ -axis is down the slope, with the origin at the snowboarder's starting point.



To find the resultant force, we first need to find the acceleration, which we can do using the equation  $a = d^2x/dt^2$ .

The acceleration  $a$  down the slope (in  $\text{m s}^{-2}$ ) is given by

$$a = \frac{d^2x}{dt^2} = \frac{d^2}{dt^2} (25(t - 1 + e^{-t})) = \frac{d}{dt} (25(1 - e^{-t})) = 25e^{-t}.$$

Now apply Newton's second law.

The mass of the snowboarder is 70 kg, so  $m = 70$ .

Hence the resultant force  $F$  (in newtons) acting on the snowboarder in the downwards direction of the slope is given in terms of the time  $t$  by

$$F = ma = 70 \times 25e^{-t} = 1750e^{-t}.$$

Here is a similar example for you to try.

**Activity 12** *Determining force from position*

A kite of mass 0.2 kg is launched from the ground and moves upwards along a straight line at an angle of  $45^\circ$  to the ground. Its distance  $x$  (in metres) from its launch point is given in terms of the time  $t$  (in seconds) after launch by

$$x = 50 \left( 1 - \frac{25}{t + 25} \right).$$

Find the magnitude of the resultant force acting on the kite at the time of launch.



Flying a kite

In both Example 7 and Activity 12 we knew information about the motion of an object, and we used this information to find the resultant force acting on the object, in terms of time. However, in a practical problem you are more likely to know the resultant force acting on an object, and want to use that to find information about the consequent motion. The next example demonstrates how to deal with this type of problem.

**Example 8** *Using Newton's second law to predict motion*

A box of mass 200 kg is pushed in a straight line along a horizontal floor. The resultant force acting on the box has constant magnitude 150 N.

If the box starts from rest and is pushed for 4 seconds, how far does it move in those 4 seconds?

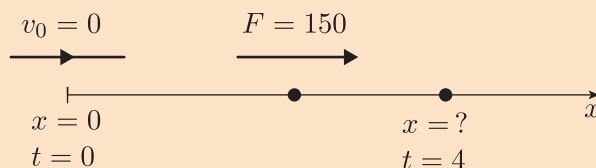
**Solution**

State any modelling assumptions. Choose when to measure time from. Draw a diagram, choose the position of the  $x$ -axis and include relevant information.

Model the box as a particle.

Measure time  $t$  from the moment the pushing starts.

Choose the positive direction of the  $x$ -axis to be the direction of the force, with the origin at the location of the box at time  $t = 0$ .



First use Newton's second law to find the acceleration of the box.

The mass of the box is 200 kg, so  $m = 200$ , and the resultant force is 150 N, so  $F = 150$ . Hence, by Newton's second law, the acceleration  $a$  (in  $\text{m s}^{-2}$ ) of the box is given by

$$a = \frac{F}{m} = \frac{150}{200} = \frac{3}{4}.$$

Now calculate the distance that the box moves. Since  $a$  is a constant, we can use the equations of motion for constant acceleration.

We know that the acceleration of the box is  $a = \frac{3}{4}$  and its initial velocity is  $v_0 = 0$ . We want to find its position  $x$  at time  $t = 4$ . Using the equation  $x = v_0 t + \frac{1}{2} a t^2$  gives

$$x = 0 \times 4 + \frac{1}{2} \times \frac{3}{4} \times 4^2 = 6.$$

So the box moves 6 m during the 4 seconds.

In Example 8, the resultant force acting on the object was constant, so the acceleration was also constant, which allowed us to use an equation of motion for constant acceleration. In the next activity, the resultant force acting on the object varies with time, and hence so does the acceleration. So you cannot use the equations of motion for constant acceleration. Instead, as in Example 7, you need to use the facts that velocity is the derivative of position with respect to time, and acceleration is the derivative of velocity with respect to time.

### Activity 13 Using Newton's second law to predict motion

A child's toy of mass 2 kg is pulled along the floor in a straight line, starting from rest. During the first 3 seconds of the pulling, the resultant force  $F$  (in newtons) acting on the toy is given by

$$F = \frac{1}{2}(1 + t),$$

where  $t$  is the time (in seconds) since the pulling began.

How far does the toy move in the first 3 seconds?

Hint: start by finding the acceleration, then the velocity, then the position, in terms of time.



A wheeled toy

In the next activity, you are given a formula for the force on an object in terms of the *position* of the object, rather than in terms of time. To solve

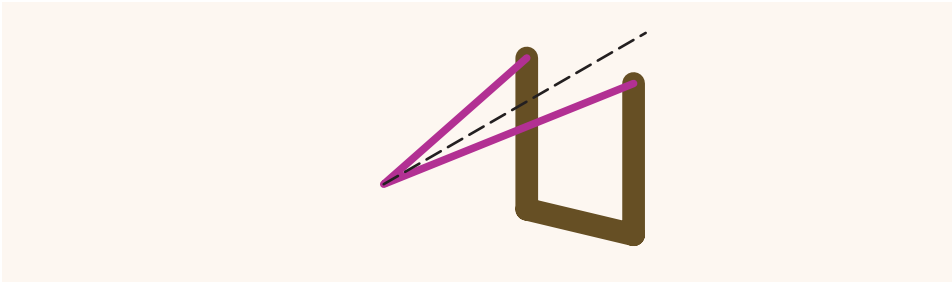
the problem in this activity you will need to use the equation

$$a = v \frac{dv}{dx},$$

and solve the resulting differential equation, as in Example 6 near the end of Subsection 1.6, as well as using Newton's second law.

### Activity 14 *Working with force as a function of position*

A model aeroplane of mass 200 g is launched by being placed in the elastic band of a catapult launcher, pulled back, and then released.



After the aeroplane is released, it moves along a straight line, as shown in the diagram, continually pushed by a force exerted by the elastic band and hence continually accelerating, until it has moved a distance of 0.4 metres. At this point the elastic band becomes slack and the aeroplane leaves the launcher and starts to fly.

During the period that the aeroplane is in contact with the elastic band, the resultant force  $F$  (in newtons) acting on the aeroplane is given by

$$F = 80 - 200x,$$

where  $x$  is the distance (in metres) of the aeroplane from its point of release.

Find, to two significant figures, the speed of the aeroplane at the moment it leaves the launcher.

Finally in this subsection, note that in fact you met an instance of Newton's second law in Unit 5. You saw there that the mass  $m$  and the weight  $\mathbf{W}$  of an object are related by the equation

$$|\mathbf{W}| = mg,$$

where  $g$  is the magnitude of the acceleration due to gravity. Since both the weight  $\mathbf{W}$  and the acceleration due to gravity are vectors that point vertically downwards, the vector version of this equation is

$$\mathbf{W} = m\mathbf{g},$$

where  $\mathbf{g}$  is the acceleration due to gravity. This equation is a particular instance of the equation  $\mathbf{F} = m\mathbf{a}$  in Newton's second law, with  $\mathbf{F} = \mathbf{W}$  and  $\mathbf{a} = \mathbf{g}$ .

## 2.2 Finding the resultant force

In the final two activities in the previous subsection, you were given a formula for the resultant force acting on an object, and you had to use Newton's second law to find the consequent acceleration of the object, and hence other details of its motion. In practical problems, usually you do not start out by knowing the resultant force acting on an object. Instead you need to find the resultant force by using information about the individual forces acting on the object, which might include weights and normal reactions. You can find the resultant force from the individual forces by using methods similar to those that you learned in Unit 5.

The individual forces acting on an object can point in several different directions, even if the resulting motion is along a straight line. Because of this, in this subsection and in the remainder of this unit we will return to using vector notation for forces, rather than representing them by the scalars that are their  $x$ -components, as we did in the previous subsection. All the problems that you will meet involve forces in two dimensions, and we will represent such forces by their component forms in terms of  $\mathbf{i}$  and  $\mathbf{j}$ , the Cartesian unit vectors in the  $x$ - and  $y$ -directions, respectively.

Remember that when you type a letter that represents a vector you should use a bold font, and when you hand write it you should underline it. So, for example, you should hand write a force  $\mathbf{F}$ , an acceleration  $\mathbf{a}$ , and the Cartesian unit vectors  $\mathbf{i}$  and  $\mathbf{j}$  as F, a, i and j, respectively.

To solve dynamics problems involving several forces, you can use the strategy below. This strategy is similar to the one that you used in Unit 5 to solve statics problems. It is a guide rather than a rigid set of rules, so you may need to adapt it for a particular problem.

It may not be clear to you at the moment exactly what each step involves, and why it is useful, but these things should become clearer when you read through the next example, which demonstrates the strategy.



**Strategy:****To solve a dynamics problem involving several forces**

1. Make any necessary modelling assumptions.
2. Draw a diagram of the physical situation, annotating it with any relevant information.
3. Choose the positions of the coordinate axes and draw them on the diagram.  
  
If the motion is along a straight line, then choose one coordinate axis (usually the  $x$ -axis) to lie along the line of motion. Usually, choose the origin to be at the initial location of the moving object, if possible.
4. Identify all the forces acting and draw a separate force diagram, stating any known magnitudes of forces. Mark the directions of  $\mathbf{i}$  and  $\mathbf{j}$  on this diagram.
5. Express the forces in component form, in terms of unknown quantities if necessary, and find the resultant force.
6. Express the acceleration in component form, in terms of unknown components.
7. Apply Newton's second law.
8. Resolve the resulting vector equation to obtain scalar equations.
9. Solve these equations to find the acceleration.
10. Hence find any other required details about the motion.
11. State a conclusion.

This strategy is demonstrated in the next example.

Note that all the examples and activities in this subsection involve objects moving across *smooth* surfaces; that is, surfaces that we model as frictionless. We will include friction in our models in the next subsection.

---

**Example 9** *Finding a resultant force, then applying Newton's second law*

A curling stone of mass 15 kg rests on smooth horizontal ice. It is pushed by a force of magnitude 26 N, acting at an angle of  $60^\circ$  to the horizontal, which causes the stone to move along the ice.

Find the magnitude of the acceleration of the stone.



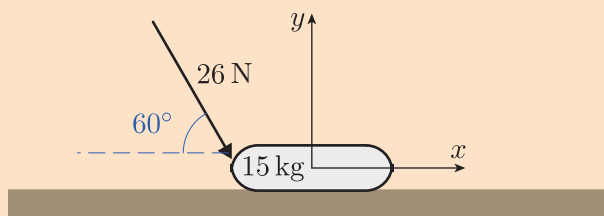
**Solution**

☁ State any modelling assumptions. ☁

Model the stone as a particle.

☁ Draw a diagram of the situation, annotating it with relevant information. Choose and draw the positions of the coordinate axes. ☁

A diagram of the situation is shown below.

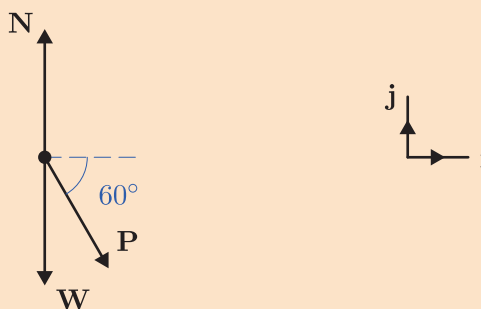


Take the  $x$ -axis to be horizontal, in the direction of the motion, and the  $y$ -axis to be vertical, with the origin at the initial location of the stone.

☁ Identify all the forces acting, and draw a separate force diagram. (Since the ice is smooth, there is no friction force.) State any known magnitudes of forces. ☁

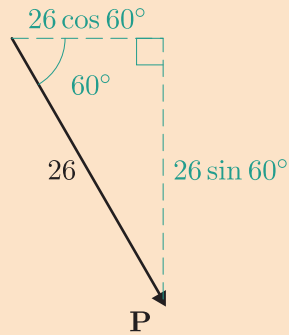
The forces acting on the stone are its weight  $\mathbf{W}$ , the normal reaction  $\mathbf{N}$  from the ice and the pushing force, say  $\mathbf{P}$ .

A force diagram is shown below.



We know that  $|\mathbf{P}| = 26$  and  $|\mathbf{W}| = 15g$ , where  $g$  is the magnitude of the acceleration due to gravity.

Express each force in component form. Draw a suitable right-angled triangle to help find the component form of  $\mathbf{P}$ , if needed. (It can be drawn on the force diagram, as in Unit 5.)



Let  $N = |\mathbf{N}|$ . Expressing the forces in component form gives

$$\mathbf{N} = N\mathbf{j}$$

$$\begin{aligned}\mathbf{P} &= 26 \cos 60^\circ \mathbf{i} - 26 \sin 60^\circ \mathbf{j} \\ &= 13\mathbf{i} - 13\sqrt{3}\mathbf{j}\end{aligned}$$

$$\mathbf{W} = -15g\mathbf{j}.$$

Find the resultant force.

The resultant force  $\mathbf{F}$  acting on the stone is given by

$$\begin{aligned}\mathbf{F} &= \mathbf{N} + \mathbf{P} + \mathbf{W} \\ &= N\mathbf{j} + 13\mathbf{i} - 13\sqrt{3}\mathbf{j} - 15g\mathbf{j} \\ &= 13\mathbf{i} + (N - 13\sqrt{3} - 15g)\mathbf{j}.\end{aligned}$$

Express the acceleration in component form, in terms of an unknown component.

Since the stone moves along the  $x$ -axis, its acceleration  $\mathbf{a}$  has component form

$$\mathbf{a} = a\mathbf{i},$$

where  $a$  is the  $\mathbf{i}$ -component of the acceleration.

Apply Newton's second law,  $\mathbf{F} = m\mathbf{a}$ . Since we want to find  $\mathbf{a}$ , it is helpful to use the law in the form  $m\mathbf{a} = \mathbf{F}$ .

Newton's second law gives

$$15a\mathbf{i} = 13\mathbf{i} + (N - 13\sqrt{3} - 15g)\mathbf{j}.$$

Resolve this vector equation to obtain scalar equations, and hence find the acceleration. Here we need resolve it only in the **i**-direction.

Resolving this equation in the **i**-direction gives

$$15a = 13,$$

so

$$a = \frac{13}{15} = 0.866 \dots$$

State a conclusion.

The magnitude of the acceleration of the stone is  $0.87 \text{ m s}^{-2}$  (to 2 s.f.).

In Example 9 the vector equation resulting from Newton's second law was resolved in the **i**-direction. It can of course also be resolved in the **j**-direction, which gives

$$0 = N - 13\sqrt{3} - 15g.$$

Solving this equation gives the value of  $N$ , the magnitude of the normal reaction, but the question did not require this quantity to be found.

In fact, since the stone in Example 9 accelerates only in the **i**-direction, it is clear from the start of the problem that the **j**-components of the forces acting on the stone must have sum zero, and hence that the resultant force acting on the stone is a force that acts in the **i**-direction and whose magnitude is the **i**-component of the pushing force. You can use this fact to find the magnitude of the acceleration of the stone. However Example 9 demonstrates the general strategy that can be used to solve more complicated problems, such as those in the next subsection.

Here is a similar activity for you to try. In this activity, once you have found the acceleration, which is constant, you can use one of the equations of motion for constant acceleration along a straight line to find the required distance.

### Activity 15 *Finding a resultant force, then applying Newton's second law*

An ice hockey puck of mass 170 g rests on smooth ice. It is pushed for half a second with a force of magnitude 0.5 N acting at an angle of  $30^\circ$  to the horizontal.

Find, to two significant figures, the magnitude of the acceleration of the puck while it is being pushed, and how far it moves while being pushed.

In Example 9 and Activity 15 the motion was in a horizontal direction. In the next example the motion is down a straight slope.

In problems like this, where the line of motion is sloping, you should normally still choose the  $x$ -axis to lie along the line of motion. This usually makes the working easier.

**Example 10** *Finding a resultant force, then applying Newton's second law, with a sloping line of motion*

A skier slides under gravity alone down a straight, smooth ski slope, which is inclined at an angle of  $30^\circ$  to the horizontal.

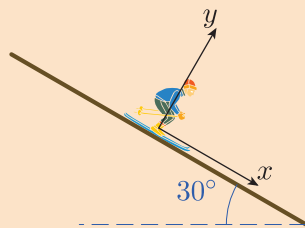
Find the magnitude of the skier's acceleration, to two significant figures.

**Solution**

State any modelling assumptions and draw a diagram of the situation. Since the skier moves down the slope, take the  $x$ -axis to point in this direction.

Model the skier as a particle.

A diagram of the situation is shown below.

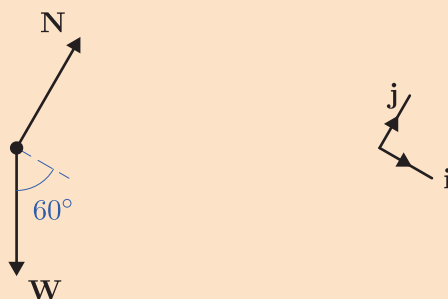


Take the  $x$ -axis to point down the slope, and the  $y$ -axis to be perpendicular to the slope.

Identify the forces acting, and draw a force diagram.

The forces acting are the weight  $\mathbf{W}$  of the skier and the normal reaction  $\mathbf{N}$  of the slope on the skier. A force diagram is shown below.

The acute angle between  $\mathbf{W}$  and the  $\mathbf{i}$ -direction is  $180^\circ - 90^\circ - 30^\circ = 60^\circ$ .



Express each force in component form, in terms of unknown quantities where necessary.

Let  $m$  be the mass of the skier; then  $|\mathbf{W}| = mg$ , where  $g$  is the magnitude of the acceleration due to gravity. Also, let  $N = |\mathbf{N}|$ .

Writing the forces in component form gives

$$\mathbf{N} = N \mathbf{j}$$

$$\begin{aligned}\mathbf{W} &= mg \cos 60^\circ \mathbf{i} - mg \sin 60^\circ \mathbf{j} \\ &= \frac{1}{2}mg \mathbf{i} - \frac{\sqrt{3}}{2}mg \mathbf{j}.\end{aligned}$$

Find the resultant force.

The resultant force  $\mathbf{F}$  is given by

$$\begin{aligned}\mathbf{F} &= \mathbf{N} + \mathbf{W} \\ &= N \mathbf{j} + \frac{1}{2}mg \mathbf{i} - \frac{\sqrt{3}}{2}mg \mathbf{j} \\ &= \frac{1}{2}mg \mathbf{i} + \left( N - \frac{\sqrt{3}}{2}mg \right) \mathbf{j}.\end{aligned}$$

Express the acceleration in component form, in terms of an unknown component.

Since the skier moves along the  $x$ -axis, her acceleration  $\mathbf{a}$  has component form

$$\mathbf{a} = a \mathbf{i},$$

where  $a$  is the  $\mathbf{i}$ -component of the acceleration.

Apply Newton's second law.

Newton's second law gives

$$ma \mathbf{i} = \frac{1}{2}mg \mathbf{i} + \left( N - \frac{\sqrt{3}}{2}mg \right) \mathbf{j}.$$

Resolve this equation in the  $\mathbf{i}$ -direction, and solve the resulting scalar equation to find the acceleration.

Resolving in the  $\mathbf{i}$ -direction gives

$$ma = \frac{1}{2}mg,$$

so

$$a = \frac{1}{2}g = \frac{1}{2} \times 9.8 = 4.9.$$

State a conclusion.

So the magnitude of the skier's acceleration is  $4.9 \text{ m s}^{-2}$  (to 2 s.f.).

Notice that in Example 10 the mass  $m$  of the skier was cancelled out from both sides of the equation obtained near the end of the working. So the acceleration of the skier does not depend on her mass, just as the acceleration of any object falling *vertically* under gravity does not depend on its mass.

Here is a similar activity for you to try.

**Activity 16** *Finding a resultant force, then applying Newton's second law, with a sloping line of motion*

A crate of mass 20 kilograms is being hauled with a rope up a smooth ramp that makes an angle of  $45^\circ$  with the horizontal. The person hauling the crate briefly holds it at rest when it is 2 metres up the ramp, but at that moment the rope suddenly breaks.

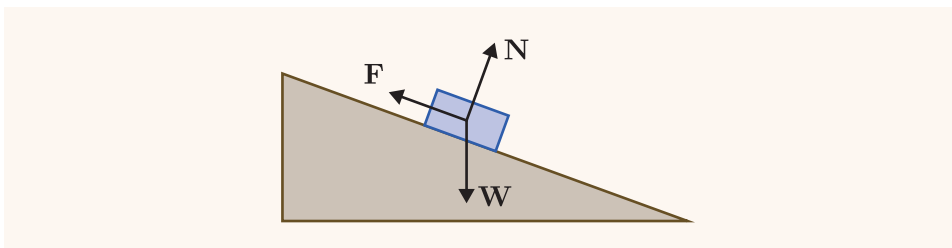
How fast will the crate be moving when it hits the bottom of the ramp?  
Give your answer to two significant figures.

You may have noticed that in Example 10 and Activity 16, where an object was sliding down an inclined plane subject only to the forces of gravity and the normal reaction of the plane on the object, the acceleration down the plane is simply the  $\mathbf{i}$ -component of the acceleration due to gravity (where  $\mathbf{i}$  points down the plane). In more complicated situations, such as those that you will see in the next subsection, the calculation of the acceleration down the plane is not so straightforward.

## 2.3 Motion with sliding friction

In all the examples and activities in Subsection 2.2, the moving object moved across a *smooth* surface – that is, we assumed that there was no friction acting. In this subsection we will consider how friction affects objects moving across rough surfaces.

You saw in Unit 5 that when an object is *at rest* on a rough surface, the force of *static friction* acts to oppose any possible movement. So, for example, the block resting on the rough inclined plane in Figure 14 will not slide down the slope, unless the angle of the slope is too large, because the force of static friction  $\mathbf{F}$  acts *up* the slope.



**Figure 14** The forces acting on a block at rest on a rough slope. Here  $\mathbf{F}$  is the static friction force,  $\mathbf{W}$  is the weight and  $\mathbf{N}$  is the normal reaction.

As you saw in Unit 5, for an object at rest on a rough surface, the magnitude of the static friction force acting on the object adjusts to balance the other forces acting on the object, up to some limit. You saw that the maximum magnitude of the static friction force is  $\mu|\mathbf{N}|$ , where  $\mu$  is a constant called the *coefficient of static friction*, which depends on the properties of the materials in contact, and  $\mathbf{N}$  is the normal reaction of the surface on the object.

If an object is *moving* across a rough surface, then it is acted on by a **sliding friction force**, which is similar to a static friction force. Sliding friction is also known as **dynamic friction** or **kinetic friction**.

Like a static friction force, a sliding friction force acts to oppose motion, though unlike a static friction force its magnitude does not adjust to balance other forces in any way. Experiments show that the magnitude of the sliding friction force on an object can be modelled by the expression  $\mu|\mathbf{N}|$ , where  $\mathbf{N}$  is the normal reaction of the surface on the object, and  $\mu$  is a constant called the **coefficient of sliding friction**, whose value depends on the properties of the materials in contact. The coefficient of sliding friction is also known as the *coefficient of dynamic friction* and the *coefficient of kinetic friction*. Note that we will use the same symbol,  $\mu$ , for both the coefficient of static friction and the coefficient of sliding friction. It should be clear from the context which is meant.

The coefficient of sliding friction between two materials is always less than the coefficient of static friction between the same two materials. This accords with our everyday experience that it is harder to start objects moving than it is to keep them moving. Some typical values of the coefficient of sliding friction are given in Table 1.

**Table 1**   Typical values of the coefficient  $\mu$  of sliding friction

| Materials in contact            | $\mu$ |
|---------------------------------|-------|
| Waxed ski on dry snow           | 0.03  |
| Brass on ice                    | 0.02  |
| Vulcanised rubber on dry tarmac | 1.07  |
| Vulcanised rubber on wet tarmac | 0.95  |

The main properties of sliding friction are summarised below.

**Sliding friction**

If an object is moving across a flat rough surface, then the friction force  $\mathbf{F}$  on the object acts parallel to the surface, in the opposite direction to the motion. Its magnitude is given by

$$|\mathbf{F}| = \mu|\mathbf{N}|,$$

where  $\mu$  is the coefficient of sliding friction for the object and the surface involved, and  $\mathbf{N}$  is the normal reaction of the surface on the object.



The next example demonstrates how to take account of sliding friction in dynamics problems. It involves the skier from Example 10 again, this time skiing over wet snow that provides some friction.

In this example, and in the activities that follow, it is necessary to resolve the vector equation resulting from Newton's second law in the  $\mathbf{j}$ -direction as well as in the  $\mathbf{i}$ -direction, unlike in the problems in the previous subsection. This is because we have to use the equation resulting from resolving in the  $\mathbf{j}$ -direction to determine the magnitude of the normal reaction, to enable us to work out the magnitude of the friction force.

### Example 11 Working with sliding friction

A skier slides under gravity down a straight ski slope covered in wet snow, which is inclined at an angle of  $30^\circ$  to the horizontal. The coefficient of sliding friction between the skis and the snow is  $\mu = 0.04$ .

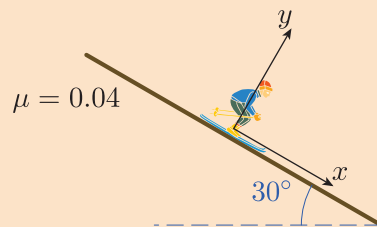
Find the magnitude of the skier's acceleration, to two significant figures.

#### Solution

State any modelling assumptions and draw a diagram of the situation. Since the skier moves down the slope, take the  $x$ -axis to point in that direction.

Model the skier as a particle.

A diagram of the situation is shown below.

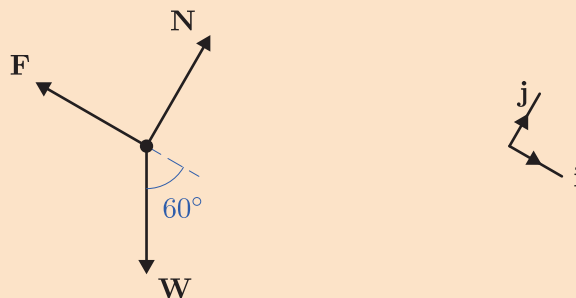


Take the  $x$ -axis to point down the slope, and the  $y$ -axis to be perpendicular to the slope, as shown.

Identify the forces acting, and draw a force diagram. State any known magnitudes of forces.

The forces acting are the weight  $\mathbf{W}$  of the skier, the normal reaction  $\mathbf{N}$  of the slope on the skier and the sliding friction force  $\mathbf{F}$ , which acts up the slope. A force diagram is shown below.





Express each force in component form, in terms of unknown quantities where necessary.

Let  $m$  be the mass of the skier; then  $|\mathbf{W}| = mg$ , where  $g$  is the magnitude of the acceleration due to gravity. Also, let  $N = |\mathbf{N}|$ . We know that  $|\mathbf{F}| = \mu|\mathbf{N}|$ , so this gives  $|\mathbf{F}| = \mu N$ .

Writing the forces in component form gives

$$\mathbf{N} = N \mathbf{j}$$

$$\mathbf{F} = -\mu N \mathbf{i}$$

$$\begin{aligned}\mathbf{W} &= mg \cos 60^\circ \mathbf{i} - mg \sin 60^\circ \mathbf{j} \\ &= \frac{1}{2}mg \mathbf{i} - \frac{\sqrt{3}}{2}mg \mathbf{j}.\end{aligned}$$

Find the resultant force. Since we have already used the symbol  $\mathbf{F}$  for the friction force, denote the resultant force by  $\mathbf{R}$ .

The resultant force  $\mathbf{R}$  is given by

$$\begin{aligned}\mathbf{R} &= \mathbf{N} + \mathbf{F} + \mathbf{W} \\ &= N \mathbf{j} - \mu N \mathbf{i} + \frac{1}{2}mg \mathbf{i} - \frac{\sqrt{3}}{2}mg \mathbf{j} \\ &= \left(\frac{1}{2}mg - \mu N\right) \mathbf{i} + \left(N - \frac{\sqrt{3}}{2}mg\right) \mathbf{j}.\end{aligned}$$

Express the acceleration in component form, in terms of an unknown component.

Since the skier moves along the  $x$ -axis, her acceleration  $\mathbf{a}$  has component form

$$\mathbf{a} = a \mathbf{i},$$

where  $a$  is the  $\mathbf{i}$ -component of the acceleration.

Apply Newton's second law.

Newton's second law gives

$$ma \mathbf{i} = \left(\frac{1}{2}mg - \mu N\right) \mathbf{i} + \left(N - \frac{\sqrt{3}}{2}mg\right) \mathbf{j}.$$

 Resolve this vector equation to obtain two scalar equations. 

Resolving this equation in the **i**- and **j**-directions gives

$$ma = \frac{1}{2}mg - \mu N$$

$$0 = N - \frac{\sqrt{3}}{2}mg.$$

 Solve these equations to find  $a$ . To do this, first find  $N$ . 

Rearranging the second equation gives

$$N = \frac{\sqrt{3}}{2}mg.$$

Substituting into the first equation gives

$$ma = \frac{1}{2}mg - \frac{\sqrt{3}}{2}\mu mg,$$

which gives

$$a = \frac{1}{2}g(1 - \mu\sqrt{3}).$$

Using  $g = 9.8$  and  $\mu = 0.04$  gives

$$a = \frac{1}{2} \times 9.8 \times (1 - 0.04 \times \sqrt{3}) = 4.56 \dots$$

 State a conclusion. 

So the magnitude of the skier's acceleration is  $4.6 \text{ m s}^{-2}$  (to 2 s.f.).

Notice that in the example above the mass  $m$  of the skier is cancelled out from both sides of an equation near the end of the working, as in Example 10. So, even with friction acting, the acceleration of the skier does not depend on her mass. Notice also that the magnitude of the skier's acceleration found in the example above is smaller than the magnitude of the acceleration found in Example 10. This is as you would expect, since friction reduces the magnitude of the acceleration. If the effect of friction in the example above is removed by taking  $\mu = 0$ , then we obtain the same value for the magnitude of the acceleration as in Example 10.

Here are two activities for you to try.

### Activity 17 Working with sliding friction

The driver of a car travelling at  $13 \text{ m s}^{-1}$  suddenly applies the brakes, causing the car to skid, in a straight line. The coefficient of sliding friction between the tyres (which are not rotating during the skid) and the road is 0.95.

Find, to two significant figures, the distance travelled by the car from the moment the brakes are applied to the moment that it comes to a halt.

**Activity 18** Working with sliding friction, again

A box is released from rest at the top of a rough inclined plane of length 2 m inclined at an angle of  $20^\circ$  to the horizontal. The coefficient of sliding friction between the box and the plane is 0.15.

Calculate, to two significant figures, the time taken for the box to slide down the plane.

## 3 Motion in two or three dimensions

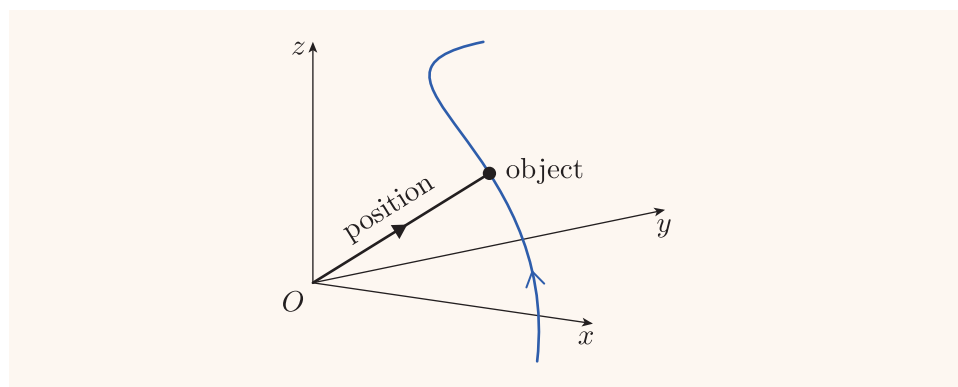
Until now all the motion that we have considered in this unit has been along a straight line. We now generalise the ideas that you have seen to apply to motion in two and three dimensions.

### 3.1 Position, velocity and acceleration in two or three dimensions

In Sections 1 and 2 you worked with position, velocity and acceleration for motion along a straight line. In this subsection you will see how to work with these quantities for motion along a curve. As earlier, we will model all moving objects as particles.

#### Position

Near the start of this unit you saw that the **position** of a moving object at any instant in time is its *position vector* with reference to some coordinate system that we have chosen, as illustrated in Figure 15. So the position of an object is its displacement from the origin.



**Figure 15** The position of an object moving along a path

When we are working with motion in two or three dimensions, we usually denote the position of the moving object by the symbol  $\mathbf{r}$ . Remember that, since  $\mathbf{r}$  is a vector, you should type it in bold, as  $\mathbf{r}$ , and hand write it underlined, as  $\underline{r}$ . You should do similarly for all the vector quantities that you work with.

We represent the position  $\mathbf{r}$  of an object by its component form in terms of the Cartesian unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$ . That is, in two dimensions we write

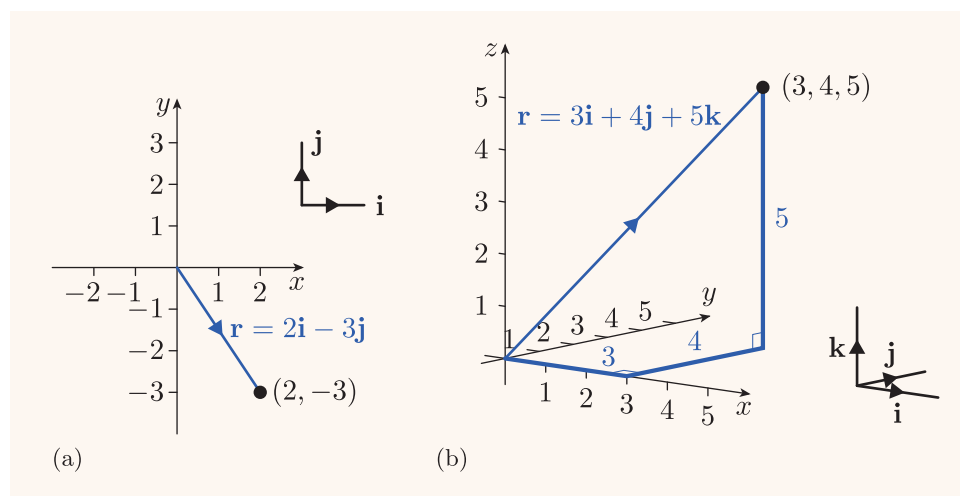
$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

and in three dimensions we write

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

where  $x$ ,  $y$  and  $z$  are the  $\mathbf{i}$ -,  $\mathbf{j}$ - and  $\mathbf{k}$ -components of  $\mathbf{r}$ , respectively.

For example, Figure 16(a) shows a particle at position  $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j}$  in two dimensions, and Figure 16(b) shows a particle at position  $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$  in three dimensions. Remember that the components of the position vector of a point are the same as the coordinates of the point, so a particle at position  $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j}$  is at the point  $(2, -3)$ , and a particle at position  $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$  is at the point  $(3, 4, 5)$ , as also illustrated in Figure 16.



**Figure 16** The positions of two objects

In a practical example, the position  $\mathbf{r}$  of a particle is measured in units of length, such as metres, and in particular its components are measured in these units.

Since the position  $\mathbf{r}$  of a moving object changes with the time  $t$ , it is a vector function of the time  $t$ , and its individual components  $x$ ,  $y$  and  $z$  are all scalar functions of the time  $t$ . (Remember that a **vector function** is a function whose output values are vectors. Similarly, a **scalar function** is a function whose output values are scalars.)

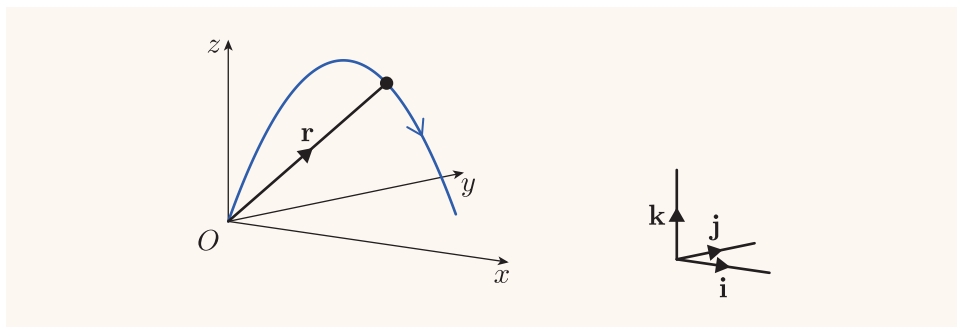
For example, the components  $x$ ,  $y$  and  $z$  of the position  $\mathbf{r}$  of a moving particle might be given in terms of the time  $t$  by

$$x = 0.3t, \quad y = 0.4t, \quad z = 12t - 5t^2,$$

which gives

$$\mathbf{r} = 0.3t\mathbf{i} + 0.4t\mathbf{j} + (12t - 5t^2)\mathbf{k}.$$

As the time  $t$  changes, the tip of the position vector  $\mathbf{r}$  traces out the path of the moving object, as illustrated in Figure 17.



**Figure 17** The path of a moving object

In the next activity, remember that the magnitude of any vector  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$  is given by  $|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$ .

### Activity 19 Calculating the position of a particle at different times

The position of a particle is given in terms of the time  $t$  by

$$\mathbf{r} = 0.3t\mathbf{i} + 0.4t\mathbf{j} + (12t - 5t^2)\mathbf{k}.$$

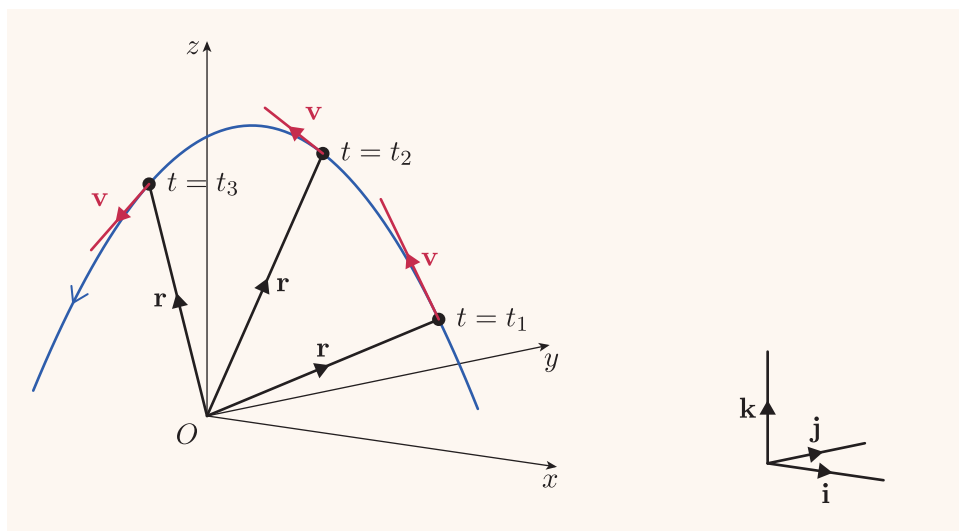
Find the position of the particle and its distance from the origin at each of the following times. Give the distance to two decimal places.

- (a)  $t = 0$       (b)  $t = 1$       (c)  $t = 2$

## Velocity

You have seen that the **velocity** of an object is the rate of change of its position with respect to time. Since position is a vector quantity, velocity is also a vector quantity, and for motion in two or three dimensions we usually denote it by  $\mathbf{v}$ . As you have seen, the magnitude of the velocity of an object is the speed with which the object is moving, and the direction of the velocity is the direction in which the object is moving. The velocity of an object can change with time, so it is a vector function of time.

This is illustrated in Figure 18, which shows the position  $\mathbf{r}$  and the velocity  $\mathbf{v}$  of a particle at some instants in time as the particle moves along a curved path. Note that both the magnitude and the direction of the velocity may change as the particle moves. The velocity vectors are always tangent to the path.



**Figure 18** The position  $\mathbf{r}$  and velocity  $\mathbf{v}$  of a particle at three instants in time

Since the velocity  $\mathbf{v}$  of a particle is the rate of change of the position  $\mathbf{r}$  of the particle with respect to time, we use the notation  $\mathbf{v} = d\mathbf{r}/dt$ .

If you know the position of a particle in terms of time, then you can use the result in the box below to find the velocity of the particle in terms of time. This result can be proved by applying the idea of differentiation from first principles to a vector function, but the details are not included here.

### Determining velocity from position

If the position of a particle is given by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

then its velocity is given by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}.$$

So, to find the velocity of a particle, you simply differentiate the components of its position with respect to time. In fact, the result in the box above extends to *any* vector function; that is, you can find the derivative of any vector function simply by differentiating each of its components separately.

The result above is given for three dimensions, but an analogous result holds for two dimensions, since the result for two dimensions is obtained by taking  $z = 0$  in the result for three dimensions. All the other results stated for three dimensions in this section also hold for two dimensions (and also for one dimension) in a similar way, and this fact is not pointed out individually in each case.

**Example 12** *Determining velocity from position*

The position of a particle is given in terms of the time  $t$  by

$$\mathbf{r} = 0.3t \mathbf{i} + 0.4t \mathbf{j} + (12t - 5t^2) \mathbf{k}.$$

Find the velocity of the particle in terms of the time  $t$ , and the speed of the particle when  $t = 1$ . Give the speed to three significant figures.

**Solution**

 The velocity is the derivative of the position with respect to time. 

The velocity is given by

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} \\ &= \frac{d}{dt}(0.3t) \mathbf{i} + \frac{d}{dt}(0.4t) \mathbf{j} + \frac{d}{dt}(12t - 5t^2) \mathbf{k} \\ &= 0.3 \mathbf{i} + 0.4 \mathbf{j} + (12 - 10t) \mathbf{k}. \end{aligned}$$

 The speed when  $t = 1$  is the magnitude of the velocity when  $t = 1$ . 

When  $t = 1$ , the velocity is

$$\begin{aligned} \mathbf{v} &= 0.3 \mathbf{i} + 0.4 \mathbf{j} + (12 - 10 \times 1) \mathbf{k} \\ &= 0.3 \mathbf{i} + 0.4 \mathbf{j} + 2 \mathbf{k}, \end{aligned}$$

so the speed is

$$\begin{aligned} |\mathbf{v}| &= \sqrt{0.3^2 + 0.4^2 + 2^2} \\ &= \sqrt{4.25} \\ &= 2.06 \text{ (to 3 s.f.)}. \end{aligned}$$

**Activity 20** *Determining velocity from position*

Each of parts (a) and (b) below gives an equation for the position of a particle in terms of time. In each case calculate the velocity and the speed of the particle at time  $t = 1$ . Give your answers for the speeds to two significant figures.

(a)  $\mathbf{r} = 2t \mathbf{i} - 3t^2 \mathbf{j}$

(b)  $\mathbf{r} = \sin 4t \mathbf{i} + \cos 4t \mathbf{j} + 3t \mathbf{k}$

You have seen that you can find the velocity of a particle in terms of time by differentiating the components of its position with respect to time. It



follows that you can find the position of a particle in terms of time by integrating the components of its velocity in terms of time, as stated in the box below.

### Determining position from velocity

If the velocity of a particle is given by

$$\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k},$$

then its position is given by

$$\begin{aligned} \mathbf{r} &= \int \mathbf{v} \, dt \\ &= \left( \int v_x \, dt \right) \mathbf{i} + \left( \int v_y \, dt \right) \mathbf{j} + \left( \int v_z \, dt \right) \mathbf{k}. \end{aligned}$$

The result in this box extends to any vector function: that is, you can integrate any vector function by integrating each of its components separately. Notice also from the box that if  $\mathbf{v}$  is a vector function of  $t$ , then we denote the result of integrating  $\mathbf{v}$  with respect to  $t$  by  $\int \mathbf{v} \, dt$ .

When you carry out an integration of the type in the box above, you obtain three scalar constants of integration, one multiplied by  $\mathbf{i}$ , one multiplied by  $\mathbf{j}$  and one multiplied by  $\mathbf{k}$ . For example, if you integrate the velocity function

$$\mathbf{v} = t \mathbf{i} + 4 \mathbf{j} - t^2 \mathbf{k}$$

then you obtain the position function

$$\begin{aligned} \mathbf{r} &= \left( \int t \, dt \right) \mathbf{i} + \left( \int 4 \, dt \right) \mathbf{j} - \left( \int t^2 \, dt \right) \mathbf{k} \\ &= \left( \frac{1}{2}t^2 + c_x \right) \mathbf{i} + (4t + c_y) \mathbf{j} + \left( -\frac{1}{3}t^3 + c_z \right) \mathbf{k}, \end{aligned}$$

where  $c_x$ ,  $c_y$  and  $c_z$  are scalar constants of integration.

However, you can combine all the scalar constants of integration into a single vector constant of integration. For example, you can write the expression for  $\mathbf{r}$  in the particular case above as

$$\begin{aligned} \mathbf{r} &= \frac{1}{2}t^2 \mathbf{i} + 4t \mathbf{j} - \frac{1}{3}t^3 \mathbf{k} + (c_x \mathbf{i} + c_y \mathbf{j} + c_z \mathbf{k}) \\ &= \frac{1}{2}t^2 \mathbf{i} + 4t \mathbf{j} - \frac{1}{3}t^3 \mathbf{k} + \mathbf{c}, \end{aligned}$$

where  $\mathbf{c} = c_x \mathbf{i} + c_y \mathbf{j} + c_z \mathbf{k}$  is a vector constant.

This is a more convenient way of dealing with the constants of integration. From now on, whenever we integrate a vector function, we will include one vector constant of integration, usually denoted by  $\mathbf{c}$ , rather than individual scalar constants of integration.

When you use integration to determine an equation for the position of an object in terms of time, as in the box above, you can find the value of the vector constant of integration  $\mathbf{c}$  if you know the position of the object at a particular time; that is, if you know an *initial condition*.



Usually the motion of a car is tightly constrained by the road direction, but not on a skid pan!

Here is an example to illustrate all the ideas above. This example involves motion in two dimensions.

### Example 13 Determining position from velocity

A car is being driven (erratically!) on a plane surface, with velocity  $\mathbf{v}$  given by

$$\mathbf{v} = 10\pi \sin \pi t \mathbf{i} + 6\pi \cos 2\pi t \mathbf{j}.$$

The position of the car at time  $t = 0$  is  $-5\mathbf{i} + 3\mathbf{j}$ . Find its position at time  $t = 1$ .

#### Solution

To find the position in terms of time, integrate the velocity with respect to time.

The position is given by

$$\begin{aligned} \mathbf{r} &= \int \mathbf{v} \, dt \\ &= \int (10\pi \sin \pi t \mathbf{i} + 6\pi \cos 2\pi t \mathbf{j}) \, dt \\ &= -10 \cos \pi t \mathbf{i} + 3 \sin 2\pi t \mathbf{j} + \mathbf{c}, \end{aligned}$$

where  $\mathbf{c}$  is a vector constant.

Use the given initial condition to find the vector constant  $\mathbf{c}$ .

We know that  $\mathbf{r} = -5\mathbf{i} + 3\mathbf{j}$  when  $t = 0$ . Substituting these values into the equation for  $\mathbf{r}$  gives

$$\begin{aligned} -5\mathbf{i} + 3\mathbf{j} &= -10 \cos 0 \mathbf{i} + 3 \sin 0 \mathbf{j} + \mathbf{c} \\ &= -10\mathbf{i} + \mathbf{c}. \end{aligned}$$

So

$$\begin{aligned} \mathbf{c} &= -5\mathbf{i} + 3\mathbf{j} + 10\mathbf{i} \\ &= 5\mathbf{i} + 3\mathbf{j}. \end{aligned}$$

Hence the position of the car is given by

$$\begin{aligned} \mathbf{r} &= -10 \cos \pi t \mathbf{i} + 3 \sin 2\pi t \mathbf{j} + 5\mathbf{i} + 3\mathbf{j} \\ &= (5 - 10 \cos \pi t) \mathbf{i} + (3 + 3 \sin 2\pi t) \mathbf{j}. \end{aligned}$$

To find the position at time  $t = 1$ , substitute  $t = 1$  into the equation for  $\mathbf{r}$ .

Hence the position of the car at time  $t = 1$  is

$$\begin{aligned} \mathbf{r} &= (5 - 10 \cos \pi) \mathbf{i} + (3 + 3 \sin 2\pi) \mathbf{j} \\ &= (5 + 10) \mathbf{i} + (3 + 0) \mathbf{j} \\ &= 15\mathbf{i} + 3\mathbf{j}. \end{aligned}$$

**Activity 21** *Determining position from velocity*

A particle moves in two-dimensional space with velocity given by

$$\mathbf{v} = 4t\mathbf{i} + 3t^2\mathbf{j}.$$

Its position at time  $t = 1$  is  $2\mathbf{i} + 3\mathbf{j}$ . What is its position, and its distance from the origin, at time  $t = 2$ ? Give the distance to two decimal places.

**Acceleration**

As you have seen, the **acceleration** of an object is the rate of change of its velocity with respect to time. Since velocity is a vector quantity, acceleration is also a vector quantity, and for motion in two or three dimensions we usually denote it by  $\mathbf{a}$ . The acceleration of an object can change with time, so it is a vector function of time.

Because the acceleration of an object is the rate of change of its velocity, an object is accelerating if either its speed is changing or if the direction of its velocity is changing, or both. In particular, an object that is moving with a constant speed but with a changing direction is accelerating. For example, an object that is moving with a constant speed along a circular path is accelerating. So the meaning of the word ‘accelerate’ in mathematics and science is slightly different from its meaning in everyday English, in which an object is said to accelerate only if its speed is changing.

You can determine the acceleration of a moving object from its velocity or position by using the facts below.

**Determining acceleration from velocity or position**

If the velocity of a particle is given by

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k},$$

then its acceleration is given by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k}.$$

If the position of a particle is given by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

then its acceleration is given by

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}.$$

You can determine the velocity of a moving object from its acceleration by using the fact in the next box.



From the Earth, geosynchronous satellites appear not to move. Although their speed is constant, they are constantly accelerating.

**Determining velocity from acceleration**

If the acceleration of a particle is given by

$$\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k},$$

then its velocity is given by

$$\begin{aligned}\mathbf{v} &= \int \mathbf{a} \, dt \\ &= \left( \int a_x \, dt \right) \mathbf{i} + \left( \int a_y \, dt \right) \mathbf{j} + \left( \int a_z \, dt \right) \mathbf{k}.\end{aligned}$$

Remember that the magnitude of position is *distance*, the magnitude of velocity is *speed*, and the magnitude of acceleration is also called *acceleration*. The intended meaning of the word ‘acceleration’ is usually clear from the context.

Here are some examples and activities to illustrate the ideas that you have met about acceleration in two and three dimensions.

**Example 14** *Determining acceleration from position*

The position of a particle is given in terms of the time  $t$  by

$$\mathbf{r} = 0.3t \mathbf{i} + 0.4t \mathbf{j} + (12t - 5t^2) \mathbf{k}.$$

Find the acceleration of the particle in terms of the time  $t$ .

**Solution**

 Differentiate the position with respect to time to obtain the velocity (this was done in Example 12), then differentiate the velocity with respect to time to obtain the acceleration. 

The velocity is given by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = 0.3\mathbf{i} + 0.4\mathbf{j} + (12 - 10t)\mathbf{k}.$$

The acceleration is given by

$$\begin{aligned}\mathbf{a} &= \frac{d\mathbf{v}}{dt} \\ &= \frac{d}{dt}(0.3) \mathbf{i} + \frac{d}{dt}(0.4) \mathbf{j} + \frac{d}{dt}(12 - 10t) \mathbf{k} \\ &= -10\mathbf{k}.\end{aligned}$$

The solution to Example 14 shows that the acceleration in this example is constant, with magnitude  $a = 10$  and direction the negative  $\mathbf{k}$ -direction.

**Activity 22** *Determining acceleration from position*

The position of a particle is given in terms of the time  $t$  by

$$\mathbf{r} = \sin t \mathbf{i} + \cos 2t \mathbf{j} + t \mathbf{k}.$$

Find the acceleration of the particle in terms of the time  $t$ .

**Example 15** *Determining position from acceleration*

The acceleration of a particle is given in terms of the time  $t$  by

$$\mathbf{a} = 18t \mathbf{i} - 20 \mathbf{j}.$$

Initially, at time  $t = 0$ , the particle has position  $\mathbf{r} = 7\mathbf{i}$  and velocity  $\mathbf{v} = 3\mathbf{j}$ . Find the position of the particle at time  $t = 10$ .

**Solution**

 To find the velocity, integrate the acceleration with respect to time. 

The velocity is given by

$$\begin{aligned} \mathbf{v} &= \int \mathbf{a} \, dt \\ &= \int (18t \mathbf{i} - 20 \mathbf{j}) \, dt \\ &= \left( \int 18t \, dt \right) \mathbf{i} - \left( \int 20 \, dt \right) \mathbf{j} \\ &= 9t^2 \mathbf{i} - 20t \mathbf{j} + \mathbf{c}, \end{aligned}$$

where  $\mathbf{c}$  is a vector constant.

 Use the initial condition for velocity to find the vector constant  $\mathbf{c}$ . 

We know that  $\mathbf{v} = 3\mathbf{j}$  when  $t = 0$ . Substituting these values into the equation above gives

$$3\mathbf{j} = 9 \times 0^2 \mathbf{i} - 20 \times 0 \mathbf{j} + \mathbf{c},$$

so  $\mathbf{c} = 3\mathbf{j}$ . This gives

$$\begin{aligned} \mathbf{v} &= 9t^2 \mathbf{i} - 20t \mathbf{j} + 3\mathbf{j} \\ &= 9t^2 \mathbf{i} + (3 - 20t) \mathbf{j}. \end{aligned}$$

 To find the position, integrate the velocity with respect to time. 

The position is given by

$$\begin{aligned}
 \mathbf{r} &= \int \mathbf{v} \, dt \\
 &= \int (9t^2 \mathbf{i} + (3 - 20t) \mathbf{j}) \, dt \\
 &= \left( \int 9t^2 \, dt \right) \mathbf{i} + \left( \int (3 - 20t) \, dt \right) \mathbf{j} \\
 &= 3t^3 \mathbf{i} + (3t - 10t^2) \mathbf{j} + \mathbf{d},
 \end{aligned}$$

where  $\mathbf{d}$  is a vector constant.

 Use the initial condition for position to find the vector constant  $\mathbf{d}$ . 

We know that  $\mathbf{r} = 7\mathbf{i}$  when  $t = 0$ . Substituting these values into the equation above gives

$$7\mathbf{i} = 3 \times 0^3 \mathbf{i} + (3 \times 0 - 10 \times 0^2) \mathbf{j} + \mathbf{d},$$

so  $\mathbf{d} = 7\mathbf{i}$ . This gives

$$\begin{aligned}
 \mathbf{r} &= 3t^3 \mathbf{i} + (3t - 10t^2) \mathbf{j} + 7\mathbf{i} \\
 &= (3t^3 + 7) \mathbf{i} + (3t - 10t^2) \mathbf{j}.
 \end{aligned}$$

 Use this formula to find the position at time  $t = 10$ . 

The position of the particle at time  $t = 10$  is

$$\begin{aligned}
 \mathbf{r} &= (3 \times 10^3 + 7) \mathbf{i} + (3 \times 10 - 10 \times 10^2) \mathbf{j} \\
 &= 3007 \mathbf{i} - 970 \mathbf{j}.
 \end{aligned}$$

### Activity 23 *Determining position from acceleration*

The acceleration of a particle is given in terms of the time  $t$  by

$$\mathbf{a} = 4e^{2t} \mathbf{i} - 2\mathbf{j}.$$

At time  $t = 0$ , the particle has position  $\mathbf{r} = 2\mathbf{j}$  and velocity  $\mathbf{v} = \mathbf{i}$ . Find the position of the particle in terms of the time  $t$ .

In this unit we have used Leibniz notation for derivatives. You can also use Lagrange notation, which you met in MST124, and there is a third notation, invented by Isaac Newton, which is often used in dynamics for derivatives with respect to time. In this notation, a dot above a variable indicates its first derivative with respect to time, and two dots above a variable indicate its second derivative with respect to time. So  $\dot{\mathbf{r}}$  denotes  $d\mathbf{r}/dt$ ,  $\dot{\mathbf{v}}$  denotes  $d\mathbf{v}/dt$  and  $\ddot{\mathbf{r}}$  denotes  $d^2\mathbf{r}/dt^2$ .

### The third derivative of position with respect to time

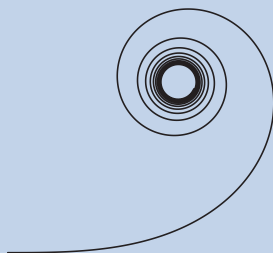
In this unit we have looked at the first and second derivatives of position with respect to time, which are velocity and acceleration, respectively. Why stop there? In some branches of engineering the third derivative of position with respect to time, known as **jerk**, is also important. So jerk is the derivative of acceleration with respect to time.

Passengers travelling in a vehicle experience non-zero jerk – that is, changing acceleration – as discomfort. For example, you experience non-zero jerk for a short time when a vehicle that you are travelling in stops suddenly.

If a train travels at a constant speed along a straight track, then its acceleration is zero, so its jerk is also zero. If it travels at a constant speed along a track that is an arc of a circle, then its acceleration is non-zero, since the direction of its velocity is constantly changing, but in fact its acceleration is constant, with a value that depends on the radius of the circular arc, so again its jerk is zero. So both of these shapes of railway track are comfortable to travel along at constant speed.

However, if a train moves directly from a circular arc to a circular arc of a different radius, or from a circular arc to a circular arc that curves in the other direction, or from a straight line to a circular arc, then its passengers will experience discomfort due to the rapidly changing acceleration – that is, the large jerk – at the transition.

To minimise this discomfort, engineers design railway tracks to have ‘transition curves’ between tracks of the shapes described above. These curves have the property that when a train travels along them at a constant speed the acceleration changes at a constant rate – that is, the jerk is constant. The transition curves are chosen to be long enough to enable the constant jerk – that is, the rate of change of the acceleration – to have a small magnitude. The curves that have the required property are called **Euler spirals**, and an example is shown below.



The magnitude of the jerk, and how it changes, are also considerations in a number of other engineering applications, including the design of automated machining tools, and the design of roller coasters.



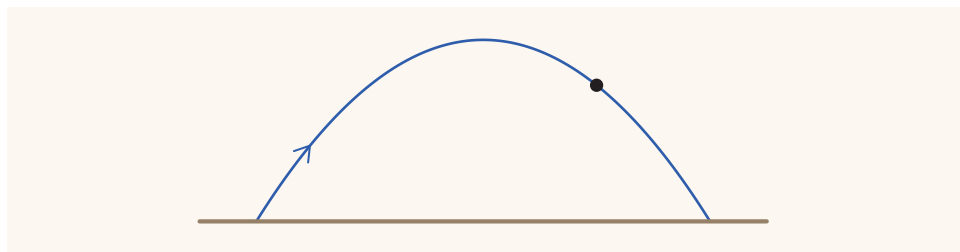
Train track designers try to minimise the jerk ...



... whereas the jerk of a roller coaster is part of the fun.

## 3.2 Projectiles

A **projectile** is an object that is propelled through space by a force that ceases after launch. The path of a projectile, as illustrated in Figure 19, is called its **trajectory**.



**Figure 19** The trajectory of a projectile

Many sports, such as basketball, golf and cricket, are based on controlling the trajectory of a projectile ball.

In this subsection you will see how you can use the ideas about position, velocity and acceleration in two dimensions that you met in the previous subsection to solve problems about projectiles.

We will model projectiles as particles, and we will ignore effects from the air on their motion. In reality, however, a projectile is an object with size as well as mass, and the air through which it travels exerts forces on it, such as air resistance, and other forces that occur if the projectile spins as it moves through the air. The effects of such forces can be analysed using more sophisticated models than we will use here.

In the first activity of this section, you can investigate the trajectories of projectiles. In particular, you are asked to consider a projectile launched from ground level over horizontal ground, and investigate which launch angle causes the projectile to travel the greatest horizontal distance before it lands on the ground. The distance along the ground travelled by a projectile is called its **range**.



### Activity 24 Investigating trajectories of projectiles

Open the *Projectiles* applet. It allows you to choose the launch speed and launch angle of a projectile, the height from which it is launched, and the angle of the ground. It shows the resulting trajectory of the projectile and its range (the distance along the ground that it travels).

Check that the launch height of the projectile and the angle of the ground are each set to zero, and that the launch speed is set to  $20 \text{ m s}^{-1}$ .



Experiment with changing the launch angle of the projectile, keeping the launch speed fixed. Which launch angle seems to give the greatest range?

Change the launch speed, and again try to find which launch angle gives the greatest range. Do the same for a third launch speed.

Later in this subsection you will be asked to prove the results that you should have found in the activity above. For now, we will start with slightly more straightforward problems about projectiles.

To model the motion of projectiles, we use the fact that any projectile is subject to a constant downwards resultant force, due to gravity. It follows, by Newton's second law, that every projectile has a constant downwards acceleration. This acceleration is the acceleration due to gravity, which, as you have seen, has magnitude  $g = 9.8 \text{ m s}^{-2}$ , to two significant figures.

In fact, you have already used this fact to solve problems about projectiles of a particular type. In Subsection 1.4 you solved problems about objects moving in a straight vertical line under the influence of gravity, such as objects that are dropped or thrown vertically upwards. These objects are projectiles launched vertically, which are sometimes called **vertical projectiles**.

The next example illustrates how you can deal with projectiles that are launched horizontally or at an angle. Since the motion of such projectiles is two-dimensional, this involves using the ideas about position, velocity and acceleration in two dimensions that you met in the last subsection.

### Example 16 *Finding the horizontal distance travelled by a projectile*

A naughty child throws a marble horizontally out of her bedroom window. When she releases the marble it is 4 m above horizontal ground, and travelling at a speed of  $2 \text{ m s}^{-1}$ .

How far does the marble travel horizontally before it hits the ground? Give your answer to two significant figures.

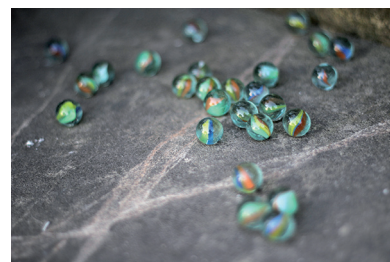
#### Solution

 Make any modelling assumptions. Choose when to measure time from. Draw a diagram of the situation, annotating it with relevant information. Choose the directions of the axes. 

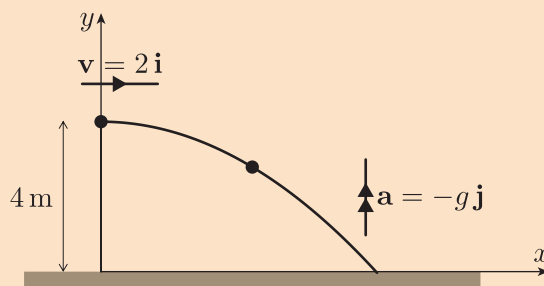
Model the marble as a particle.

Measure time from the moment that the child releases the marble.

Choose the  $x$ -axis to be horizontal, pointing in the direction of the marble's initial velocity, and choose the  $y$ -axis to point vertically upwards, with the origin at the point at ground level directly below where the child releases the marble.



Marbles are best kept on the ground



Express the known information about the position, velocity and acceleration of the marble in component form.

We know that at time  $t = 0$  the position is  $\mathbf{r} = 4\mathbf{j}$  and the velocity is  $\mathbf{v} = 2\mathbf{i}$ .

The acceleration  $\mathbf{a}$  is the acceleration due to gravity, so it has magnitude  $g$  and points downwards. This gives  $\mathbf{a} = -g\mathbf{j}$ .

Use the formula for the acceleration to determine the velocity in terms of time.

The velocity of the marble is given by

$$\begin{aligned}\mathbf{v} &= \int \mathbf{a} \, dt \\ &= - \int g\mathbf{j} \, dt \\ &= -gt\mathbf{j} + \mathbf{c},\end{aligned}$$

where  $\mathbf{c}$  is a vector constant.

At time  $t = 0$  the velocity is  $\mathbf{v} = 2\mathbf{i}$ . Substituting these values into the equation above gives

$$2\mathbf{i} = -g \times 0\mathbf{j} + \mathbf{c},$$

so  $\mathbf{c} = 2\mathbf{i}$ . This gives

$$\mathbf{v} = 2\mathbf{i} - gt\mathbf{j}.$$

 Use the formula for the velocity to determine the position in terms of time. 

The position of the marble is given by

$$\begin{aligned}\mathbf{r} &= \int \mathbf{v} \, dt \\ &= \int (2\mathbf{i} - gt\mathbf{j}) \, dt \\ &= 2t\mathbf{i} - \frac{1}{2}gt^2\mathbf{j} + \mathbf{d},\end{aligned}$$

where  $\mathbf{d}$  is a vector constant.

At time  $t = 0$  the position is  $\mathbf{r} = 4\mathbf{j}$ . Substituting these values into the equation above gives

$$4\mathbf{j} = 2 \times 0\mathbf{i} - \frac{1}{2}g \times 0^2\mathbf{j} + \mathbf{d},$$

so  $\mathbf{d} = 4\mathbf{j}$ .

Hence the position of the marble at time  $t$  is given by

$$\begin{aligned}\mathbf{r} &= 2t\mathbf{i} - \frac{1}{2}gt^2\mathbf{j} + 4\mathbf{j} \\ &= 2t\mathbf{i} + (4 - \frac{1}{2}gt^2)\mathbf{j}.\end{aligned}$$

 Now use the formula for the position to solve the problem. Here, we can use it to find the time  $t$  when the marble reaches the ground, and hence find the  $\mathbf{i}$ -component of its position at this time. 

The marble reaches the ground when the  $\mathbf{j}$ -component of its position is 0; that is, when

$$4 - \frac{1}{2}gt^2 = 0.$$

This gives

$$t^2 = \frac{8}{g}.$$

The time when the marble reaches the ground will be positive, which gives

$$t = \sqrt{\frac{8}{g}}.$$

At this time  $t$ , the  $\mathbf{i}$ -component of the position is

$$2t = 2\sqrt{\frac{8}{g}} = 2\sqrt{\frac{8}{9.8}} = 1.80 \dots$$

So the marble travels a horizontal distance of 1.8 m (to 2 s.f.) before it hits the ground.

You might like to check the answer to Example 16, and your answers to some of the activities in this subsection, by using the *Projectiles* applet. It can provide more information than you used in Activity 24.

Here is an activity similar to Example 16 for you to try. In this activity, the projectile is launched at an angle rather than horizontally, so you will need to use trigonometry to find the component form of its launch velocity. To find the maximum height that the projectile reaches, remember that when it reaches its highest point, the vertical component of its velocity will be changing from positive to negative, and hence will be zero momentarily.

### Activity 25 *Finding the horizontal distance travelled and the maximum height reached by a projectile*

A ball is kicked at an angle of  $10^\circ$  above the horizontal at a speed of  $30 \text{ m s}^{-1}$  over horizontal ground. Find the following, to two significant figures.

- (a) The distance that the ball travels horizontally before it hits the ground.
- (b) The maximum height reached by the ball.

In the next activity, you are asked to find the required launch speed of a projectile, so in modelling the situation you should denote this speed by a symbol, say  $v_0$ . You should find that once you have obtained a formula for the position of the projectile in terms of time, you can form and then resolve a suitable vector equation to obtain a pair of simultaneous equations in  $v_0$  and a relevant time  $t$ , and solve these equations to find the value of  $v_0$ .



Basketball is one of the world's most watched sports

### Activity 26 *Finding a required launch speed of a projectile*

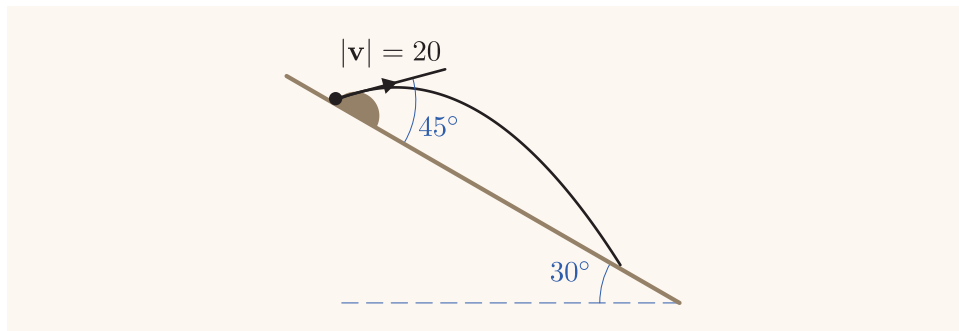
A basketball player throws a ball towards the hoop. The centre of the hoop is at a height of 3 m, and is at a horizontal distance of 1.5 m from the player. The player throws the ball from a height of 2 m above the ground, at an angle of  $65^\circ$  above the horizontal.

At what speed does the player need to throw the ball to get it through the hoop? Give your answer to two significant figures.

In the next activity, we return once again to modelling the motion of a snowboarder. In this activity the ground is not horizontal, so you should take the  $x$ -axis to be parallel to the slope of the ground, as this makes the working easier. You will need to use trigonometry to find the component form of the acceleration **a**.

**Activity 27** *Working with a projectile travelling over sloping ground*

A bump causes a snowboarder to take off from an otherwise flat slope. The slope is at  $30^\circ$  to the horizontal, and the snowboarder takes off with a speed of  $20 \text{ m s}^{-1}$  at an angle of  $45^\circ$  to the slope, as shown in the diagram below.



Calculate the time that the snowboarder spends in the air. Give your answer to two significant figures.



Many snowboarding manoeuvres are based on skateboarding tricks

In the final activity of this unit, you are asked to consider a projectile travelling over horizontal ground, launched with a general launch speed and a general launch angle. By doing this you can prove the result about the maximum range of such a projectile that you should have observed in Activity 24.

Part (a) of the activity is similar to part (a) of Activity 25, just with a general launch speed and a general launch angle.

**Activity 28** *Proving the optimal launch angle for a projectile*

A projectile is launched from ground level, over horizontal ground, with speed  $v_0$  at the angle  $\theta$  to the horizontal, where  $0^\circ \leq \theta \leq 90^\circ$ .

- (a) Show that the range of the projectile (the distance along the ground that it travels before returning to the ground) is given by

$$\frac{2v_0^2 \sin \theta \cos \theta}{g},$$

where  $g$  is the magnitude of the acceleration due to gravity.

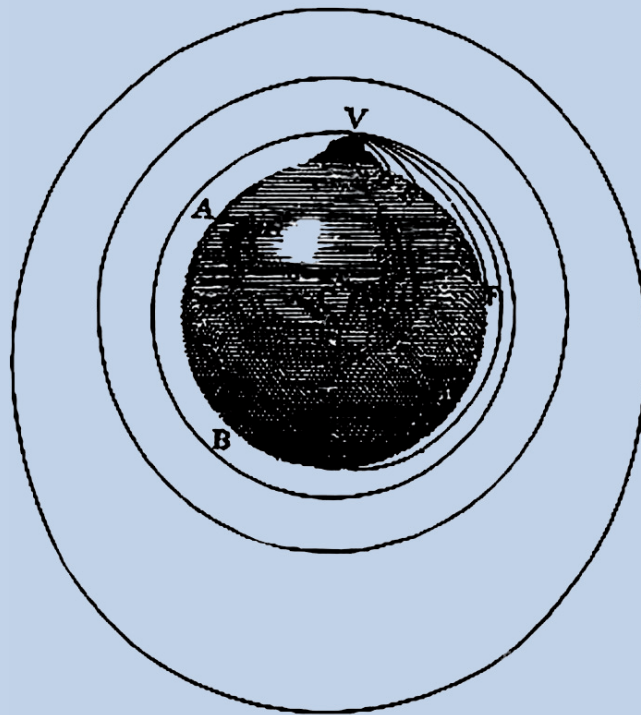
- (b) Hence find the value of  $\theta$  that gives the maximum range for any given launch speed  $v_0$ .

Hint: you may find the trigonometric identity  $\sin 2\theta = 2 \sin \theta \cos \theta$  useful.

### Projectiles and orbits

One of the many achievements of Isaac Newton in developing his three laws of motion was to produce a theory that could describe many different types of motion.

In *The System of the World*, Newton's early and less formal account of Book III of the *Principia*, which was published in 1728, a year after his death, Newton included the following diagram of a thought experiment relating the motion of projectiles to the orbits of objects around the Earth.



To understand Newton's thought experiment, imagine yourself standing on top of a large mountain (labelled *V* in the diagram), launching a projectile horizontally. For small launch speeds, the projectile will fall to Earth near the foot of the mountain. As the launch speed increases, the distance from the foot of the mountain where the projectile lands will increase. If the curvature of the Earth is taken into account, then the larger the launch speed, the further round the Earth the projectile will travel. If the launch speed is sufficiently large, then the projectile will travel all the way round the Earth and hit you on the back of the head! The projectile has gone into orbit!

In this subsection you have been solving problems involving objects subject to a *constant* resultant force, due to gravity. More generally, as a particle moves along a path in one, two or three dimensions, the resultant force on the particle can change, both in magnitude and in direction. You can solve problems involving motion like this by using Newton's second law, which you met in Section 2. It was stated as follows.

### Newton's second law of motion

If a particle of constant mass  $m$  is acted on by a resultant force  $\mathbf{F}$ , then its acceleration  $\mathbf{a}$  is given by

$$\mathbf{F} = m\mathbf{a}.$$

Newton's second law tells you that wherever a moving particle is on its path, the resultant force and the acceleration of the particle are always in the same direction, and their magnitudes are related by the equation  $|\mathbf{F}| = m|\mathbf{a}|$ . You will apply Newton's second law to solve problems involving motion in two or three dimensions if you go on to take higher-level modules in applied mathematics.

## Learning outcomes

After studying this unit you should be able to:

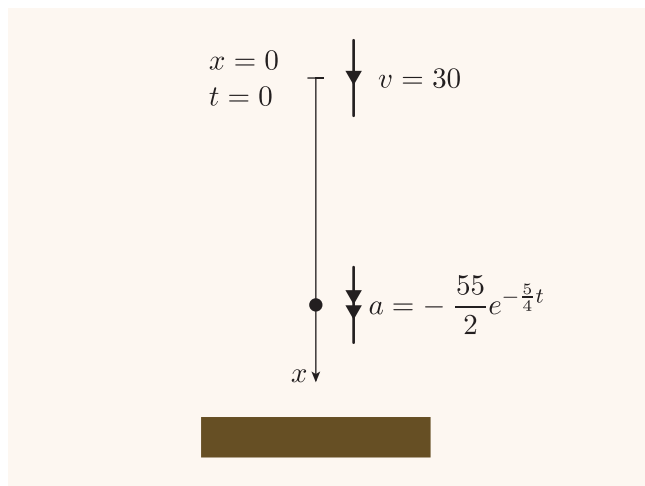
- understand and work with position, velocity, acceleration and force as vectors in one, two or three dimensions, and as represented by scalars in the case of one dimension
- obtain a formula for the position, velocity or acceleration of an object in terms of time, when you know a formula for one of the other two of these three quantities and suitable initial conditions if necessary
- obtain a formula for the velocity or acceleration of an object in terms of its position, when you know a formula for the other one of these two quantities and a suitable initial condition if necessary
- apply the five equations for motion with constant acceleration along a straight line
- apply Newton's second law to determine the acceleration of an object when you know its mass and the resultant force acting on it
- determine the resultant force acting on an object, when you know the individual forces acting on it
- solve dynamics problems involving sliding friction
- solve dynamics problems about projectiles.

# Solutions to activities

## Solution to Activity 1

Model the parachutist as a particle.

A diagram of the situation is below. We are told that  $v = 30$  when  $t = 0$ .



Choose the  $x$ -axis to point vertically down, with the origin at the parachutist's location at  $t = 0$ , so that  $x = 0$  when  $t = 0$ .

- (a) The velocity is found by integrating the acceleration. This gives

$$\begin{aligned} v &= \int a \, dt \\ &= \int \left( -\frac{55}{2} e^{-\frac{5}{4}t} \right) dt \\ &= -\frac{55}{2} \times \frac{1}{-5/4} e^{-\frac{5}{4}t} + c \\ &= 22e^{-\frac{5}{4}t} + c, \end{aligned}$$

where  $c$  is a constant.

Using the initial condition that  $v = 30$  when  $t = 0$  gives

$$30 = 22e^{-\frac{5}{4} \times 0} + c,$$

so

$$c = 30 - 22e^0 = 30 - 22 = 8.$$

So the velocity, in  $\text{ms}^{-1}$ , is given in terms of the time  $t$  by

$$v = 8 + 22e^{-\frac{5}{4}t}.$$

- (b) The position is found by integrating the velocity. This gives

$$\begin{aligned} x &= \int v \, dt \\ &= \int \left( 8 + 22e^{-\frac{5}{4}t} \right) dt \\ &= 8t + 22 \times \frac{1}{-5/4} e^{-\frac{5}{4}t} + d \\ &= 8t - \frac{88}{5} e^{-\frac{5}{4}t} + d, \end{aligned}$$

where  $d$  is a constant.

Using the initial condition that  $x = 0$  when  $t = 0$  gives

$$0 = 8 \times 0 - \frac{88}{5} e^{-\frac{5}{4} \times 0} + d,$$

so

$$d = \frac{88}{5}.$$

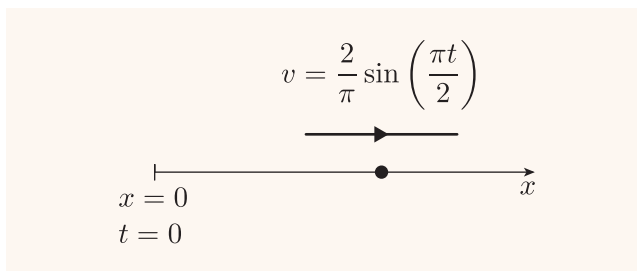
So the position, in m, is given in terms of the time  $t$  by

$$x = 8t - \frac{88}{5} e^{-\frac{5}{4}t} + \frac{88}{5}.$$

## Solution to Activity 2

Model the shuttle as a particle.

A diagram of the situation is shown below.



Since we are told that a positive velocity corresponds to motion to the right, choose the  $x$ -axis to point to the right. Choose the origin to be at the left-hand side of the loom, so that  $x = 0$  when  $t = 0$ .



- (a) The position is found by integrating the velocity. This gives

$$\begin{aligned} x &= \int v \, dt \\ &= \int \frac{2}{\pi} \sin\left(\frac{\pi t}{2}\right) dt \\ &= -\frac{4}{\pi^2} \cos\left(\frac{\pi t}{2}\right) + c, \end{aligned}$$

where  $c$  is a constant.

Using the initial condition that  $x = 0$  when  $t = 0$  gives

$$0 = -\frac{4}{\pi^2} \cos 0 + c,$$

so

$$c = \frac{4}{\pi^2} \cos 0 = \frac{4}{\pi^2}.$$

So the position, in metres, of the shuttle is given by

$$\begin{aligned} x &= -\frac{4}{\pi^2} \cos\left(\frac{\pi t}{2}\right) + \frac{4}{\pi^2} \\ &= \frac{4}{\pi^2} \left(1 - \cos\left(\frac{\pi t}{2}\right)\right). \end{aligned}$$

- (b) Since the maximum and minimum values of  $\cos\left(\frac{\pi t}{2}\right)$  are 1 and  $-1$ , respectively, the value of the position  $x$  varies between

$$x = \frac{4}{\pi^2}(1 - 1) = 0$$

and

$$x = \frac{4}{\pi^2}(1 + 1) = \frac{8}{\pi^2}.$$

So the width of the loom in metres is

$$\frac{8}{\pi^2} - 0 = \frac{8}{\pi^2} = 0.810 \dots$$

That is, the width is 81 cm, to the nearest cm.

### Solution to Activity 3

- (a) The position  $x$  of the particle increases until approximately time  $t = 2$ , after which it decreases. So the direction of motion of the particle reverses at time approximately 2 seconds.

- (b) The graph shows that the particle starts at position  $x = 0$ , approximately, and that at time  $t = 2$  its position is  $x = 1$ , approximately. So when the direction of motion of the particle reverses the particle is approximately 1 metre from its starting position.
- (c) The speed is greatest at the time when the magnitude of the gradient of the position–time graph is the greatest. This is at time approximately 2.75 seconds.

### Solution to Activity 4

- (a) The speed is greatest at the time when the magnitude of the velocity is greatest. This is at time 0 seconds.
- (b) When the velocity is positive, the particle is moving in the positive  $x$ -direction, and when the velocity is negative, the particle is moving in the negative  $x$ -direction. So the direction of motion of the particle reverses when the curve crosses the  $t$ -axis, which is at time approximately 0.25 seconds.
- (c) The change in the position of the particle from time  $t = 0$  to time  $t = 4$  is the signed area between the velocity–time graph and the  $t$ -axis from time  $t = 0$  to time  $t = 4$ . Since the area below the  $t$ -axis is larger than the area above the  $t$ -axis, the change in the position is negative. So, since the particle starts at the origin, it finishes at time  $t = 4$  with a negative position.

### Solution to Activity 5

- (a) We eliminate  $a$  from equations (1) and (2). Equation (1) can be rearranged as

$$a = \frac{v - v_0}{t},$$

and substituting this expression for  $a$  into equation (2) gives

$$\begin{aligned} x &= v_0 t + \frac{1}{2} \times \frac{v - v_0}{t} t^2 \\ &= v_0 t + \frac{(v - v_0)t}{2} \\ &= \left(v_0 + \frac{v}{2} - \frac{v_0}{2}\right)t \\ &= \frac{v_0 + v}{2}t \\ &= \frac{1}{2}(v_0 + v)t. \end{aligned}$$

- (b) We eliminate
- $v_0$
- from equations (1) and (2).

Equation (1) can be rearranged as

$$v_0 = v - at,$$

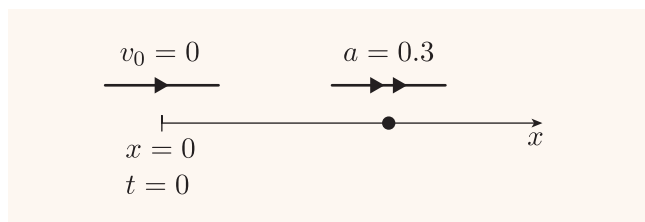
and substituting this expression for  $v_0$  into equation (2) gives

$$\begin{aligned} x &= (v - at)t + \frac{1}{2}at^2 \\ &= vt - at^2 + \frac{1}{2}at^2 \\ &= vt - \frac{1}{2}at^2. \end{aligned}$$

**Solution to Activity 6**

Model the train as a particle.

Measure time from the moment the train starts accelerating.

Choose the positive direction of the  $x$ -axis to be the direction in which the train is accelerating, with the origin at the point where the train is at time  $t = 0$ .The position of the train at time  $t$  is  $x$ . Since the train starts from rest, its initial velocity is  $v_0 = 0$ . The acceleration is  $a = 0.3$ .We consider the point in the motion when  $x = 35$ . So we know values for  $a$ ,  $x$  and  $v_0$ , and we want to find the corresponding value of the velocity  $v$ .By the equation  $v^2 = v_0^2 + 2ax$ , the velocity  $v$  of the train after it has travelled 35 m is given by

$$v^2 = 0 + 2 \times 0.3 \times 35 = 21.$$

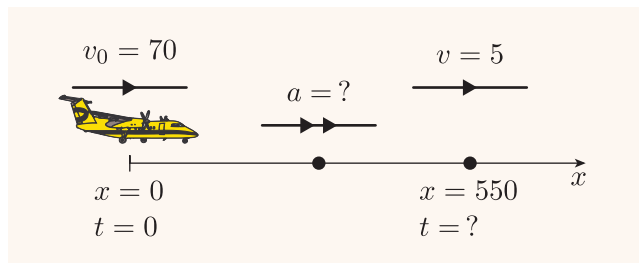
Since the velocity  $v$  is always in the positive direction of the  $x$ -axis, this gives

$$v = \sqrt{21} = 4.58 \dots$$

So the velocity of the train after it has travelled 35 m is  $4.6 \text{ m s}^{-1}$  (to 2 s.f.).**Solution to Activity 7**

Model the aircraft as a particle.

Measure time from the moment the aircraft touches down.

Choose the positive direction of the  $x$ -axis to be the direction in which the aircraft is moving, with the origin at the point of touchdown.We know that the initial velocity is  $v_0 = 70$ , and that at the end of the runway, where  $x = 550$ ,  $v = 5$ . Let the unknown acceleration of the aircraft be  $a$ .

- (a) Here we know that
- $v_0 = 70$
- , and
- $v = 5$
- when
- $x = 550$
- . We want to calculate the value of
- $a$
- . Using
- $v^2 = v_0^2 + 2ax$
- gives

$$5^2 = 70^2 + 2a \times 550,$$

so

$$\begin{aligned} a &= \frac{5^2 - 70^2}{2 \times 550} \\ &= -4.4 \text{ (to 2 s.f.)}. \end{aligned}$$

The magnitude of the acceleration needed is  $4.4 \text{ m s}^{-2}$ .

(The negative acceleration is as expected, since the velocity is positive and the aircraft is slowing down.)

- (b) Here we know that
- $v_0 = 70$
- , and
- $v = 5$
- when
- $x = 550$
- , again, and we want to calculate the corresponding value of
- $t$
- . Using
- $x = \frac{1}{2}(v_0 + v)t$
- gives

$$550 = \frac{70 + 5}{2} t.$$

Hence

$$\begin{aligned} t &= 550 \times \frac{2}{75} \\ &= 15 \text{ (to 2 s.f.)}. \end{aligned}$$

So the aircraft takes 15 seconds to travel the length of the runway.

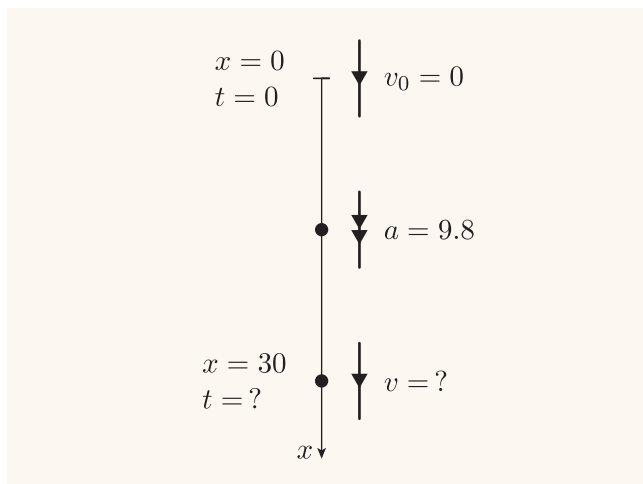
**Solution to Activity 8**

Model the stone as a particle.

Measure time from the moment the stone is released.

Choose the positive direction of the  $x$ -axis to be downwards, with the origin at the point from where the stone is released.We know that the initial velocity is  $v_0 = 0$ , that the acceleration is the acceleration due to gravity, which

has magnitude  $9.8 \text{ m s}^{-2}$  and is directed downwards, and that the stone falls 30 m.



- (a) Here we know that  $v_0 = 0$  and  $a = 9.8$ , and we want to find the time  $t$  when the position  $x$  is 30.

Using  $x = v_0 t + \frac{1}{2} a t^2$  gives

$$30 = 0 \times t + \frac{1}{2} \times 9.8 \times t^2,$$

so

$$t^2 = \frac{60}{9.8}.$$

Since the time at which the stone hits the bottom of the well must be positive,

$$t = \sqrt{\frac{60}{9.8}} = 2.474 \dots$$

So the stone hits the bottom of the well after 2.5 s (to 2 s.f.).

- (b) Here we know that  $v_0 = 0$  and  $a = 9.8$ , and we want to find the velocity  $v$  when the position  $x$  is 30.

Using  $v^2 = v_0^2 + 2ax$  gives

$$\begin{aligned} v^2 &= 0^2 + 2 \times 9.8 \times 30 \\ &= 588. \end{aligned}$$

Since the stone will be moving downwards as it hits the bottom of the well, the value of  $v$  must be positive, so

$$v = \sqrt{588} = 24.24 \dots$$

So the speed of the stone is  $24 \text{ m s}^{-1}$  (to 2 s.f.) when it hits the bottom of the well.

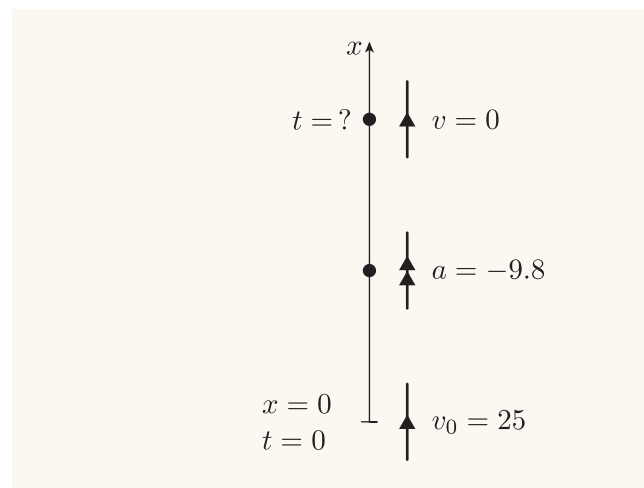
## Solution to Activity 9

Model the ball as a particle.

Measure time from the moment the ball is released.

Choose the  $x$ -axis to point vertically upwards, with the origin at the point from where the ball is released.

We know that the initial velocity of the ball is  $v_0 = 25$ , and that its acceleration  $a$  has magnitude  $g = 9.8$  and is directed vertically downwards (that is, in the opposite direction to the positive direction of the  $x$ -axis), so  $a = -9.8$ .



- (a) (i) The ball reaches its maximum height when the velocity is zero.

Here we know that  $v_0 = 25$  and  $a = -9.8$ , and we want to find the value of the position  $x$  when  $v = 0$ .

Using  $v^2 = v_0^2 + 2ax$  gives

$$0 = 25^2 + 2(-9.8)x,$$

so

$$x = \frac{25^2}{2 \times 9.8} = 31.8 \dots$$

The maximum height reached by the ball is 32 m (to 2 s.f.) above the ground.

- (ii) Here we know that  $v_0 = 25$  and  $a = -9.8$ , and we want to find the value of the time  $t$  when  $v = 0$ .

Using  $v = v_0 + at$  gives

$$0 = 25 - 9.8t, \quad \text{so} \quad t = \frac{25}{9.8} = 2.55 \dots$$

The ball reaches its maximum height 2.6 seconds (to 2 s.f.) after it is released.

- (iii) The ball returns to the ground when  $x = 0$ .

Here we know that  $v_0 = 25$  and  $a = -9.8$ , and we want to find the value of the time  $t$  when  $x = 0$ .

Using  $x = v_0t + \frac{1}{2}at^2$  gives

$$0 = 25t + \frac{1}{2}(-9.8)t^2;$$

that is,

$$0 = 25t - 4.9t^2.$$

Factorising gives

$$0 = t(25 - 4.9t).$$

So the solutions are  $t = 0$  and

$$t = \frac{25}{4.9} = 5.10 \dots$$

The time for the ball to return to the ground must be positive, so it is 5.1 s (to 2 s.f.).

(The solution  $t = 0$  tells us that  $x = 0$  also when  $t = 0$ , which we already knew!)

- (b) By the equations  $v = v_0 + at$  and  $x = v_0t + \frac{1}{2}at^2$ , the velocity and position of the ball are given in terms of the time  $t$  by the equations

$$v = 25 - 9.8t,$$

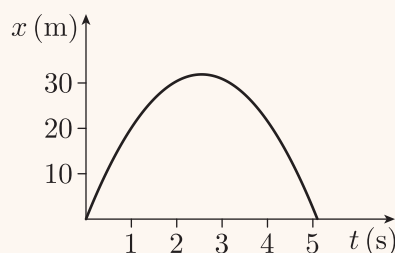
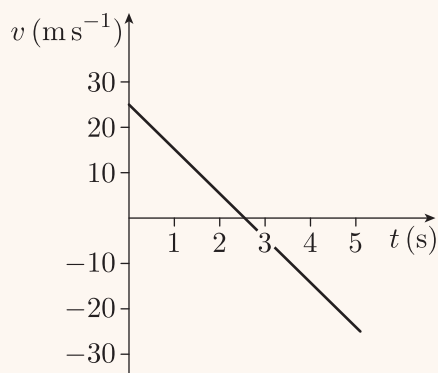
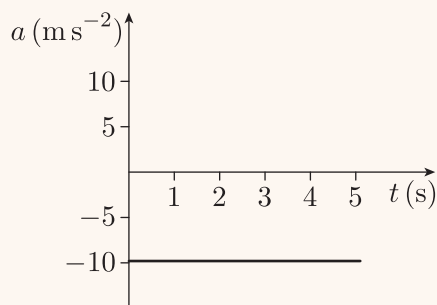
$$x = 25t - 4.9t^2.$$

We also know that  $a$  is a constant function of the time  $t$ , with

$$a = -9.8.$$

We know that the ball reaches a maximum height of 32 m at time 2.6 s, and returns to a height of 0 m at time 5.1 s.

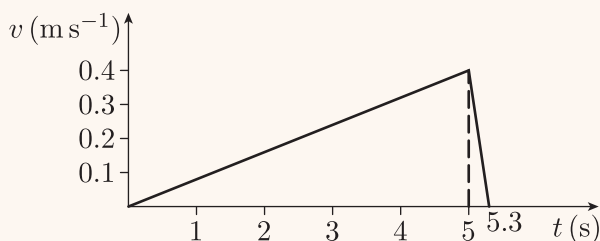
Graphs of the acceleration, velocity and position against time are shown below.



### Solution to Activity 10

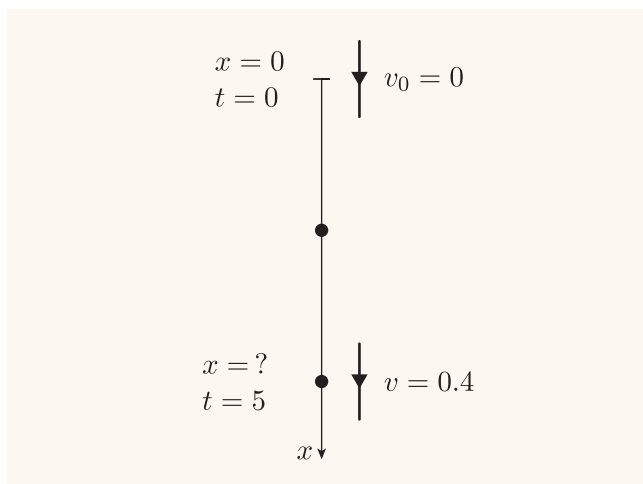
Model the yo-yo as a particle.

A velocity–time graph is shown below.



### First stage

For the speeding up stage of the motion, measure time from when the yo-yo is at its highest position, and take the  $x$ -axis to point vertically downwards, with the origin at the point where the yo-yo is at time  $t = 0$ .



We know that the initial velocity is  $v_0 = 0$  and the velocity at time  $t = 5$  is  $v = 0.4$ . We want to find the position  $x$  of the yo-yo at this time.

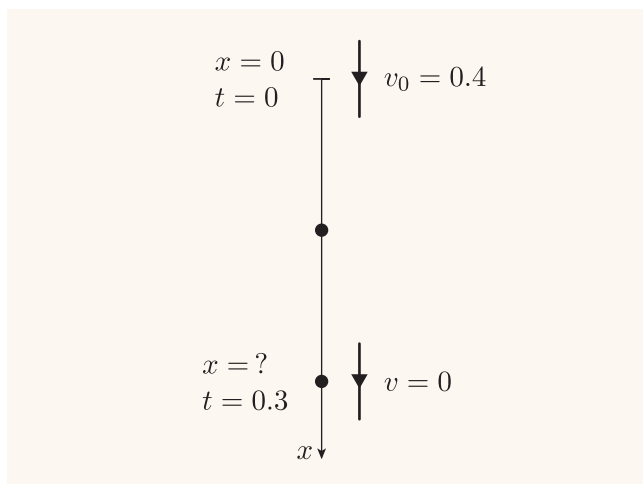
Using  $x = \frac{1}{2}(v_0 + v)t$  gives

$$x = \frac{1}{2}(0 + 0.4) \times 5 = 1.$$

So the distance travelled in this stage is 1 m.

### Second stage

For the slowing down stage of the motion, measure time from the start of this stage, and take the  $x$ -axis to point vertically downwards, with the origin at the point where the yo-yo is at the start of this stage.



We know that the initial velocity is  $v_0 = 0.4$  and the velocity at time  $t = 0.3$  is  $v = 0$ . We want to find the position  $x$  of the yo-yo at this time.

Using  $x = \frac{1}{2}(v_0 + v)t$  gives

$$x = \frac{1}{2}(0.4 + 0) \times 0.3 = 0.06.$$

So the distance travelled in this stage is 0.06 m.

### Complete motion

The total distance between the highest and lowest positions of the yo-yo is

$$\begin{aligned} 1 \text{ m} + 0.06 \text{ m} &= 1.06 \text{ m} \\ &= 1.1 \text{ m (to 2 s.f.).} \end{aligned}$$

The acceleration is constant in each of the two stages of the motion, but its magnitude is larger in the second stage, since the magnitude of the total change in the velocity is the same in the two stages, but the change happens over a shorter time period in the second stage. (That is, the magnitude of the gradient of the velocity–time graph is greater in the second stage.)

So we want to find the magnitude of the acceleration in the second stage.

### Second stage again

As before, we know that the initial velocity is  $v_0 = 0.4$  and the velocity at time  $t = 0.3$  is  $v = 0$ . We want to find the acceleration  $a$ .

Using  $v = v_0 + at$  gives

$$0 = 0.4 + a \times 0.3$$

so

$$a = -\frac{0.4}{0.3} = -1.33 \dots$$

So the maximum magnitude of the acceleration is  $1.3 \text{ m s}^{-2}$  (to 2 s.f.).

(The negative acceleration is as expected, since in the second stage the yo-yo is still travelling in the positive  $x$ -direction but is slowing down.)

## Solution to Activity 11

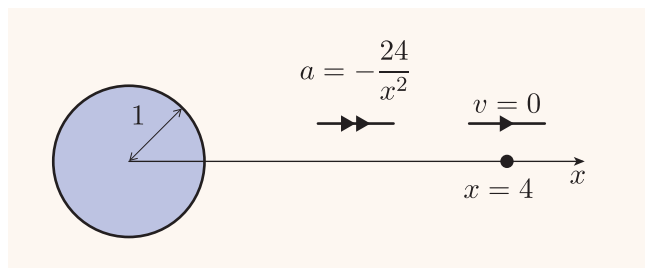
Model the ball bearing as a particle.

Take the  $x$ -axis to be as specified in the question.

We know that the velocity is  $v = 0$  when the position is  $x = 4$ .

Since the acceleration is directed in the negative  $x$ -direction,

$$a = -\frac{24}{x^2}.$$



Using the equation  $a = v \frac{dv}{dx}$  gives

$$v \frac{dv}{dx} = -\frac{24}{x^2}.$$

Separating the variables gives

$$\int v \, dv = - \int \frac{24}{x^2} \, dx,$$

which gives

$$\frac{v^2}{2} = \frac{24}{x} + c,$$

where  $c$  is a constant.

Using the initial condition that  $v = 0$  when  $x = 4$  gives

$$0 = \frac{24}{4} + c,$$

so  $c = -6$ .

Hence the velocity of the particle is given in terms of the position  $x$  by

$$\frac{v^2}{2} = \frac{24}{x} - 6;$$

that is,

$$v^2 = \frac{48}{x} - 12.$$

Since the ball bearing moves in the negative  $x$ -direction (towards the magnet), its velocity will be negative for all values of  $x$  involved. Hence the velocity is given by

$$v = -\sqrt{\frac{48}{x} - 12}.$$

The ball bearing hits the magnet when  $x = 1$ . At this point its velocity  $v$  is given by

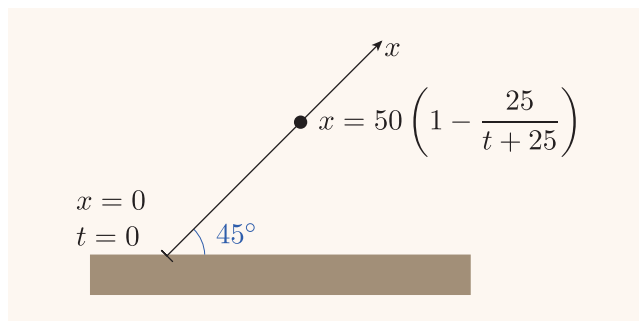
$$v = -\sqrt{48 - 12} = -\sqrt{36} = -6.$$

The speed of the ball bearing when it hits the magnet is  $6 \text{ cm s}^{-1}$ .

## Solution to Activity 12

Model the kite as a particle.

The  $x$ -axis points up at  $45^\circ$ , along the path of the kite, with the origin at the launch point.



The acceleration of the kite is given by

$$\begin{aligned} a &= \frac{d^2x}{dt^2} \\ &= \frac{d^2}{dt^2} \left( 50 \left( 1 - \frac{25}{t+25} \right) \right) \\ &= \frac{d}{dt} \left( \frac{50 \times 25}{(t+25)^2} \right) \\ &= \frac{d}{dt} \left( \frac{1250}{(t+25)^2} \right) \\ &= -\frac{2500}{(t+25)^3}. \end{aligned}$$

The mass of the kite is  $0.2 \text{ kg}$ , so  $m = 0.2$ .

Hence, by Newton's second law, the resultant force  $F$  (in newtons) on the kite is given by

$$F = ma = 0.2 \times \left( -\frac{2500}{(t+25)^3} \right) = -\frac{500}{(t+25)^3},$$

acting along the direction of motion.

At launch the time is  $t = 0$ , which gives

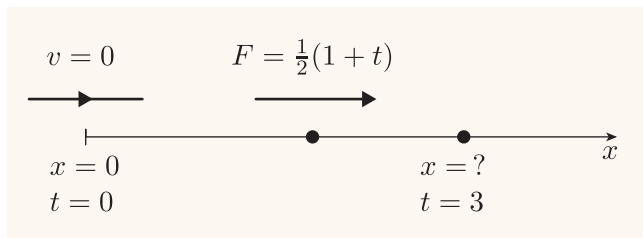
$$F = -\frac{500}{(0+25)^3} = -0.032.$$

So the magnitude of the resultant force acting on the kite at launch is  $0.032 \text{ N}$ .

### Solution to Activity 13

Model the toy as a particle.

Choose the positive direction of the  $x$ -axis to be the direction of motion, with the origin at the location of the toy at time  $t = 0$ .



First we find the acceleration  $a$  of the toy in terms of the time  $t$ .

The mass of the toy is 2 kg, so  $m = 2$ . Hence, by Newton's second law, the acceleration (in  $\text{ms}^{-2}$ ) is

$$a = \frac{F}{m} = \frac{\frac{1}{2}(1+t)}{2} = \frac{1}{4}(1+t).$$

Now we find the velocity  $v$  (in  $\text{ms}^{-1}$ ) of the toy in terms of the time  $t$ . We have

$$\begin{aligned} v &= \int a \, dt \\ &= \int \frac{1}{4}(1+t) \, dt \\ &= \frac{1}{4}\left(t + \frac{1}{2}t^2\right) + c \\ &= \frac{1}{8}(2t + t^2) + c, \end{aligned}$$

where  $c$  is a constant.

The toy starts from rest, so  $v = 0$  when  $t = 0$ .

Substituting these values into the equation above gives

$$0 = c,$$

so

$$v = \frac{1}{8}(2t + t^2).$$

Now we find the position  $x$  (in metres) of the toy in terms of the time  $t$ . We have

$$\begin{aligned} x &= \int v \, dt \\ &= \int \frac{1}{8}(2t + t^2) \, dt \\ &= \frac{1}{8}\left(t^2 + \frac{1}{3}t^3\right) + d \\ &= \frac{1}{24}(3t^2 + t^3) + d, \end{aligned}$$

where  $d$  is a constant.

Now  $x = 0$  when  $t = 0$  (because of how we positioned the  $x$ -axis). Substituting these values into the equation above gives

$$0 = d,$$

so

$$x = \frac{1}{24}(3t^2 + t^3).$$

Hence the position of the toy at time  $t = 3$  is given by

$$\begin{aligned} x &= \frac{1}{24}(3 \times 3^2 + 3^3) \\ &= \frac{1}{24} \times 54 \\ &= 2.25. \end{aligned}$$

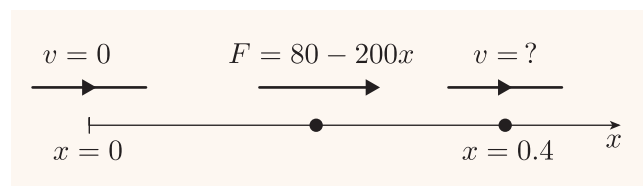
So the toy moves 2.25 m in the first 3 seconds.

### Solution to Activity 14

Model the aeroplane as a particle.

As implied in the question, take the  $x$ -axis to point in the direction of motion, with the origin at the release point of the aeroplane.

The problem is summarised in the diagram below.



The resultant force (in newtons) on the aeroplane is given by  $F = 80 - 200x$ , and the mass (in kg) of the aeroplane is given by  $m = 0.2$ . So, by Newton's second law, the acceleration  $a$  (in  $\text{ms}^{-2}$ ) of the aeroplane is given by

$$a = \frac{F}{m} = \frac{80 - 200x}{0.2} = 400 - 1000x.$$

This expresses the acceleration  $a$  as a function of the position  $x$ . To find the velocity  $v$  (in  $\text{ms}^{-1}$ ) as a function of the position  $x$ , we use the equation

$$a = v \frac{dv}{dx},$$

which gives

$$v \frac{dv}{dx} = 400 - 1000x.$$

This is a separable differential equation.

Separating the variables gives

$$\int v \, dv = \int (400 - 1000x) \, dx,$$

which gives

$$\frac{v^2}{2} = 400x - 500x^2 + c,$$

where  $c$  is a constant.

Using the initial condition that  $v = 0$  when  $x = 0$  gives  $0 = c$ . Hence

$$\frac{v^2}{2} = 400x - 500x^2;$$

that is,

$$v^2 = 800x - 1000x^2.$$

Since the aeroplane moves in the positive  $x$ -direction, its velocity  $v$  is always positive, so

$$v = \sqrt{800x - 1000x^2}.$$

The aeroplane leaves the launcher when  $x = 0.4$ , which gives

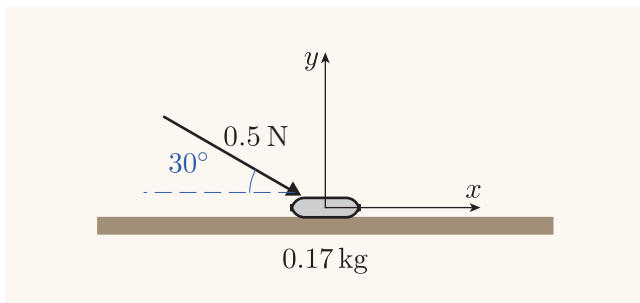
$$\begin{aligned} v &= \sqrt{800 \times 0.4 - 1000 \times 0.4^2} \\ &= \sqrt{160} \\ &= 12.6 \dots \end{aligned}$$

The speed of the aeroplane at the moment it leaves the launcher is therefore  $13 \, \text{m s}^{-1}$  (to 2 s.f.).

### Solution to Activity 15

Model the puck as a particle.

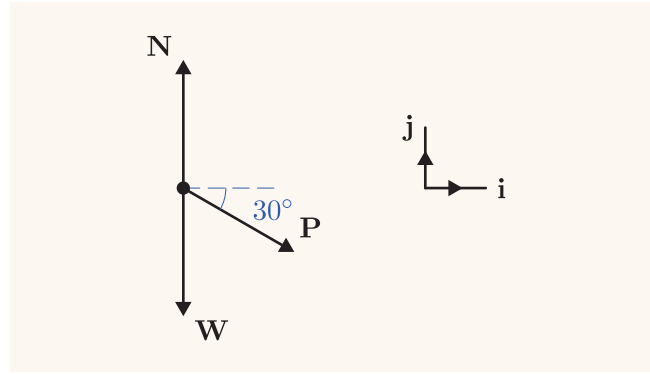
A diagram of the situation is shown below.



Take the  $x$ -axis to be horizontal, along the line of motion of the puck, and the  $y$ -axis to be vertical, with the origin at the initial location of the puck.

The forces acting on the puck are its weight  $\mathbf{W}$ , the normal reaction  $\mathbf{N}$  from the ice and the pushing force  $\mathbf{P}$ , say.

A force diagram is shown next.



We know that  $|\mathbf{W}| = 0.17g$  (since the mass of the puck is  $170 \, \text{g} = 0.17 \, \text{kg}$ ) and  $|\mathbf{P}| = 0.5$ .

Let  $N = |\mathbf{N}|$ . Expressing the forces in component form gives

$$\mathbf{N} = N \mathbf{j}$$

$$\begin{aligned} \mathbf{P} &= 0.5 \cos 30^\circ \mathbf{i} - 0.5 \sin 30^\circ \mathbf{j} \\ &= \frac{\sqrt{3}}{4} \mathbf{i} - \frac{1}{4} \mathbf{j}. \end{aligned}$$

$$\mathbf{W} = -0.17g \mathbf{j}$$

The resultant force is

$$\begin{aligned} \mathbf{F} &= \mathbf{N} + \mathbf{P} + \mathbf{W} \\ &= N \mathbf{j} + \frac{\sqrt{3}}{4} \mathbf{i} - \frac{1}{4} \mathbf{j} - 0.17g \mathbf{j} \\ &= \frac{\sqrt{3}}{4} \mathbf{i} + \left( N - \frac{1}{4} - 0.17g \right) \mathbf{j}. \end{aligned}$$

Since the puck moves along the  $x$ -axis, its acceleration  $\mathbf{a}$  has component form

$$\mathbf{a} = a \mathbf{i},$$

where  $a$  is the  $\mathbf{i}$ -component of the acceleration.

Newton's second law gives

$$0.17a \mathbf{i} = \frac{\sqrt{3}}{4} \mathbf{i} + \left( N - \frac{1}{4} - 0.17g \right) \mathbf{j}.$$

Resolving in the  $\mathbf{i}$ -direction gives

$$0.17a = \frac{\sqrt{3}}{4},$$

so

$$a = \frac{\sqrt{3}}{4 \times 0.17} = \frac{\sqrt{3}}{0.68} = 2.54 \dots$$

The magnitude of the acceleration of the puck while being pushed is therefore  $2.5 \, \text{m s}^{-2}$  (to 2 s.f.). Since the acceleration is constant, the distance moved by the puck can be found by using the equations of motion for constant acceleration.



We know that the initial velocity is  $v_0 = 0$  and the acceleration  $a$  (in  $\text{m s}^{-2}$ ) is  $\sqrt{3}/0.68$ , and we want to find the position  $x$  (in metres, measured from the starting position) at time  $t = \frac{1}{2}$  (in seconds).

Using  $x = v_0 t + \frac{1}{2} a t^2$  gives

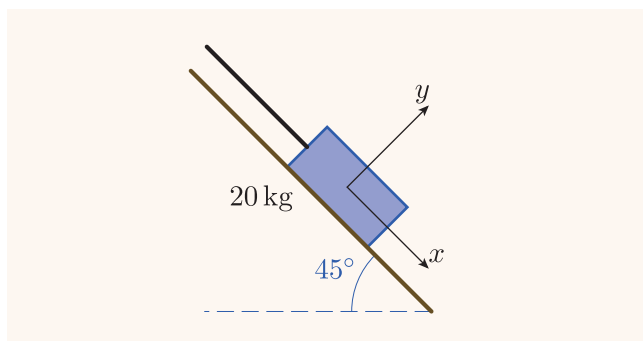
$$x = 0 \times \frac{1}{2} + \frac{1}{2} \times \frac{\sqrt{3}}{0.68} \times \left(\frac{1}{2}\right)^2 = 0.318 \dots$$

Hence the distance moved by the puck is 0.32 m (to 2 s.f.).

### Solution to Activity 16

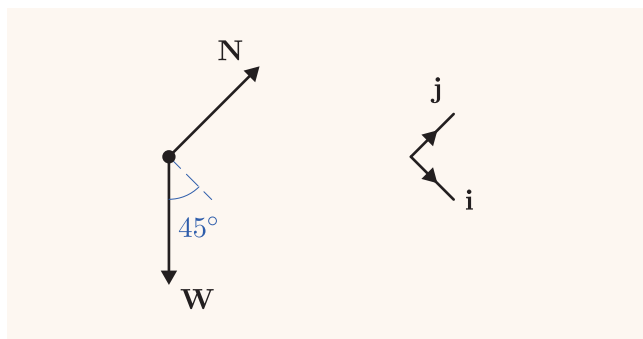
Model the crate as a particle.

A diagram of the situation is shown below.



Take the  $x$ -axis to point down the slope, and the  $y$ -axis to be perpendicular to the slope, with the origin at the location of the crate when the rope breaks.

The forces acting on the crate are its weight  $\mathbf{W}$  and the normal reaction  $\mathbf{N}$  from the ramp. A force diagram is shown below.



We know that  $|\mathbf{W}| = 20g$ . Let  $N = |\mathbf{N}|$ . Then

$$\mathbf{N} = N \mathbf{j}$$

$$\begin{aligned} \mathbf{W} &= 20g \cos 45^\circ \mathbf{i} - 20g \sin 45^\circ \mathbf{j} \\ &= 10\sqrt{2}g \mathbf{i} - 10\sqrt{2}g \mathbf{j}. \end{aligned}$$

The resultant force is

$$\begin{aligned} \mathbf{F} &= \mathbf{N} + \mathbf{W} = N \mathbf{j} + 10\sqrt{2}g \mathbf{i} - 10\sqrt{2}g \mathbf{j} \\ &= 10\sqrt{2}g \mathbf{i} + (N - 10\sqrt{2}g) \mathbf{j}. \end{aligned}$$

Since the crate moves along the  $x$ -axis, its acceleration  $\mathbf{a}$  has component form

$$\mathbf{a} = a \mathbf{i},$$

where  $a$  is the  $\mathbf{i}$ -component of the acceleration.

Newton's second law gives

$$20a \mathbf{i} = 10\sqrt{2}g \mathbf{i} + (N - 10\sqrt{2}g) \mathbf{j}.$$

Resolving in the  $\mathbf{i}$ -direction gives

$$20a = 10\sqrt{2}g,$$

so

$$a = \frac{10\sqrt{2}g}{20} = \frac{g}{\sqrt{2}}.$$

So the crate moves with a constant acceleration of  $g/\sqrt{2}$  from rest (since it is at rest when the rope breaks) for a distance of 2 m.

By the equation  $v^2 = v_0^2 + 2ax$ , the velocity  $v$  of the crate when it hits the bottom of the ramp is given by

$$v^2 = 0 + 2 \times \frac{g}{\sqrt{2}} \times 2 = 2\sqrt{2}g.$$

The velocity  $v$  of the crate at the bottom of the ramp is positive, so

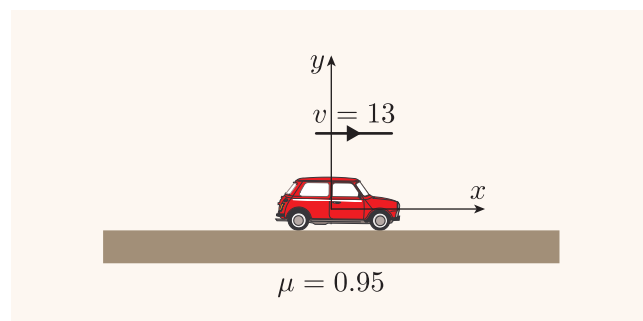
$$v = \sqrt{2\sqrt{2}g} = \sqrt{2\sqrt{2} \times 9.8} = 5.26 \dots$$

So the crate will be moving at a speed of  $5.3 \text{ m s}^{-1}$  (to 2 s.f.) when it hits the bottom of the ramp.

### Solution to Activity 17

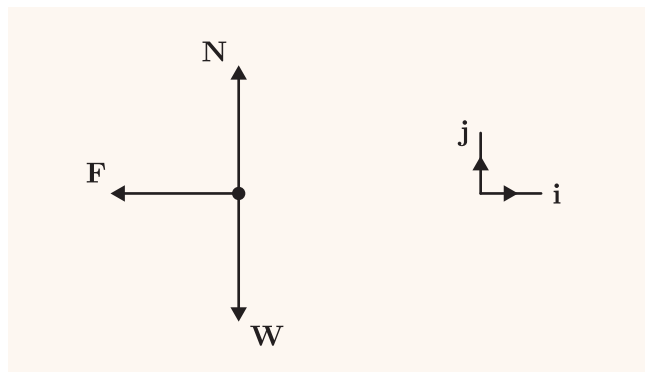
Model the car as a particle.

A diagram of the situation is shown below.



Take the  $x$ -axis to point in the direction of motion, and the  $y$ -axis to be vertical, with the origin at the location of the car when the brakes are applied, as shown.

The forces acting are the weight  $\mathbf{W}$  of the car, the normal reaction  $\mathbf{N}$  of the road on the car and the sliding friction force  $\mathbf{F}$ . A force diagram is shown below.



Let  $m$  be the mass of the car; then  $|\mathbf{W}| = mg$ , where  $g$  is the magnitude of the acceleration due to gravity. Also, let  $N = |\mathbf{N}|$ . We know that  $|\mathbf{F}| = \mu|\mathbf{N}|$ , so this gives  $|\mathbf{F}| = \mu N$ .

Writing the forces in component form gives

$$\mathbf{N} = N\mathbf{j}$$

$$\mathbf{F} = -\mu N\mathbf{i}$$

$$\mathbf{W} = -mg\mathbf{j}.$$

The resultant force  $\mathbf{R}$  is

$$\begin{aligned}\mathbf{R} &= \mathbf{N} + \mathbf{F} + \mathbf{W} \\ &= N\mathbf{j} - \mu N\mathbf{i} - mg\mathbf{j} \\ &= -\mu N\mathbf{i} + (N - mg)\mathbf{j}.\end{aligned}$$

Since the car moves along the  $x$ -axis, its acceleration  $\mathbf{a}$  has component form

$$\mathbf{a} = a\mathbf{i},$$

where  $a$  is the  $\mathbf{i}$ -component of the acceleration.

Newton's second law gives

$$ma\mathbf{i} = -\mu N\mathbf{i} + (N - mg)\mathbf{j}.$$

Resolving in the  $\mathbf{i}$ - and  $\mathbf{j}$ -directions gives

$$ma = -\mu N$$

$$0 = N - mg.$$

The second equation gives

$$N = mg.$$

Substituting in the first equation gives

$$ma = -\mu mg,$$

so

$$a = -\mu g = -0.95 \times 9.8 = -9.31.$$

So the car has initial velocity  $v_0 = 13$  (in  $\text{m s}^{-1}$ ) and constant acceleration  $a = -9.31$  (in  $\text{m s}^{-2}$ ), and we want to find the position  $x$  when the velocity  $v$  is 0.

Using the equation  $v^2 = v_0^2 + 2ax$  gives

$$0^2 = 13^2 + 2(-9.31)x,$$

which gives

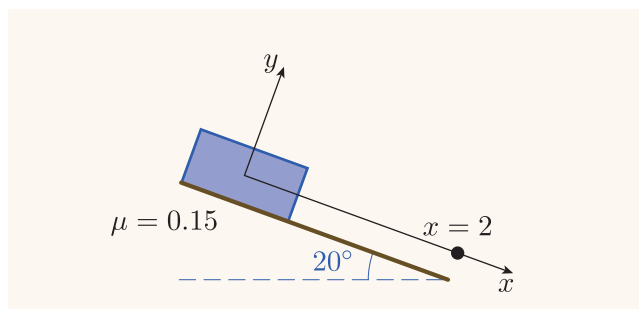
$$x = \frac{13^2}{2 \times 9.31} = 9.07 \dots$$

So the car travels 9.1 m (to 2 s.f.) before it comes to a halt.

### Solution to Activity 18

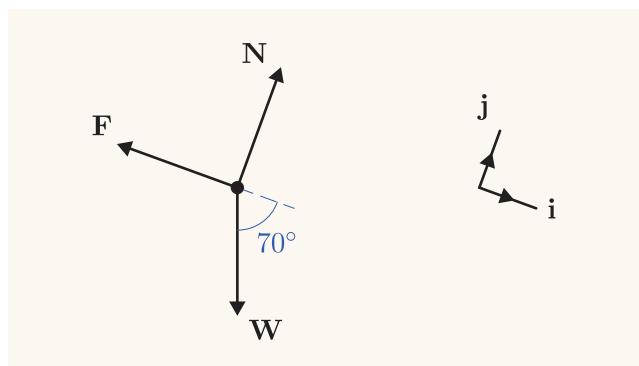
Model the box as a particle.

A diagram of the situation is shown below.



Take the  $x$ -axis to point down the slope, and the  $y$ -axis to be perpendicular to the slope, with the origin at the point from which the box is released, as shown.

The forces acting are the weight  $\mathbf{W}$  of the box, the normal reaction  $\mathbf{N}$  of the slope on the box and the sliding friction force  $\mathbf{F}$ . A force diagram is shown below.



Let  $m$  be the mass of the box; then  $|\mathbf{W}| = mg$ , where  $g$  is the magnitude of the acceleration due to gravity. Also, let  $N = |\mathbf{N}|$ . We know that  $|\mathbf{F}| = \mu|\mathbf{N}|$ , so this gives  $|\mathbf{F}| = \mu N$ .

Writing the forces in component form gives

$$\mathbf{N} = N \mathbf{j}$$

$$\mathbf{F} = -\mu N \mathbf{i}$$

$$\mathbf{W} = mg \cos 70^\circ \mathbf{i} - mg \sin 70^\circ \mathbf{j}.$$

The resultant force  $\mathbf{R}$  is

$$\mathbf{R} = \mathbf{N} + \mathbf{F} + \mathbf{W}$$

$$= N \mathbf{j} - \mu N \mathbf{i} + mg \cos 70^\circ \mathbf{i} - mg \sin 70^\circ \mathbf{j}$$

$$= (mg \cos 70^\circ - \mu N) \mathbf{i} + (N - mg \sin 70^\circ) \mathbf{j}.$$

Since the box moves along the  $x$ -axis, its acceleration has component form

$$\mathbf{a} = a \mathbf{i},$$

where  $a$  is the  $\mathbf{i}$ -component of the acceleration.

Newton's second law gives

$$ma \mathbf{i} = (mg \cos 70^\circ - \mu N) \mathbf{i} + (N - mg \sin 70^\circ) \mathbf{j}.$$

Resolving in the  $\mathbf{i}$ - and  $\mathbf{j}$ -directions gives

$$ma = mg \cos 70^\circ - \mu N$$

$$0 = N - mg \sin 70^\circ.$$

The second equation gives

$$N = mg \sin 70^\circ.$$

Substituting into the first equation gives

$$ma = mg \cos 70^\circ - \mu mg \sin 70^\circ,$$

so

$$a = g \cos 70^\circ - \mu g \sin 70^\circ$$

$$= 9.8 \cos 70^\circ - 0.15 \times 9.8 \sin 70^\circ$$

$$= 1.97 \dots$$

So the box has initial velocity  $v_0 = 0$  and constant acceleration  $a = 1.97 \dots$  (in  $\text{ms}^{-2}$ ), and we want to find the time  $t$  when the position is  $x = 2$ .

Using the equation  $x = v_0 t + \frac{1}{2} a t^2$  gives

$$2 = 0 + \frac{1}{2} a t^2,$$

which gives

$$t^2 = \frac{4}{a}.$$

Since the time  $t$  must be positive, this gives

$$t = \frac{2}{\sqrt{a}} = \frac{2}{\sqrt{1.97 \dots}} = 1.42 \dots$$

So the time taken for the box to slide down the plane is 1.4 seconds (to 2 s.f.).

## Solution to Activity 19

(a) When  $t = 0$ , the position is

$$\begin{aligned} \mathbf{r} &= (0.3 \times 0) \mathbf{i} + (0.4 \times 0) \mathbf{j} \\ &\quad + (12 \times 0 - 5 \times 0^2) \mathbf{k} \\ &= \mathbf{0}. \end{aligned}$$

The distance from the origin is

$$|\mathbf{r}| = |\mathbf{0}| = 0.$$

(b) When  $t = 1$ , the position is

$$\begin{aligned} \mathbf{r} &= (0.3 \times 1) \mathbf{i} + (0.4 \times 1) \mathbf{j} \\ &\quad + (12 \times 1 - 5 \times 1^2) \mathbf{k} \\ &= 0.3 \mathbf{i} + 0.4 \mathbf{j} + 7 \mathbf{k}. \end{aligned}$$

The distance from the origin is

$$\begin{aligned} |\mathbf{r}| &= |0.3 \mathbf{i} + 0.4 \mathbf{j} + 7 \mathbf{k}| \\ &= \sqrt{0.3^2 + 0.4^2 + 7^2} \\ &= 7.02 \text{ (to 2 d.p.)}. \end{aligned}$$

(c) When  $t = 2$ , the position is

$$\begin{aligned} \mathbf{r} &= (0.3 \times 2) \mathbf{i} + (0.4 \times 2) \mathbf{j} \\ &\quad + (12 \times 2 - 5 \times 2^2) \mathbf{k} \\ &= 0.6 \mathbf{i} + 0.8 \mathbf{j} + 4 \mathbf{k}. \end{aligned}$$

The magnitude is

$$\begin{aligned} |\mathbf{r}| &= |0.6 \mathbf{i} + 0.8 \mathbf{j} + 4 \mathbf{k}| \\ &= \sqrt{0.6^2 + 0.8^2 + 4^2} \\ &= 4.12 \text{ (to 2 d.p.)}. \end{aligned}$$

## Solution to Activity 20

(a) The velocity is given by

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} \\ &= \frac{d}{dt} (2t) \mathbf{i} - \frac{d}{dt} (3t^2) \mathbf{j} \\ &= 2 \mathbf{i} - 6t \mathbf{j}. \end{aligned}$$

So at time  $t = 1$  the velocity of the particle is

$$\mathbf{v} = 2 \mathbf{i} - 6 \mathbf{j},$$

and the speed is

$$\begin{aligned} |\mathbf{v}| &= \sqrt{4 + 36} \\ &= \sqrt{40} \\ &= 6.32 \dots \\ &= 6.3 \text{ (to 2 s.f.)}. \end{aligned}$$

(b) The velocity is given by

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} \\ &= \frac{d}{dt}(\sin 4t) \mathbf{i} + \frac{d}{dt}(\cos 4t) \mathbf{j} + \frac{d}{dt}(3t) \mathbf{k} \\ &= 4 \cos 4t \mathbf{i} - 4 \sin 4t \mathbf{j} + 3 \mathbf{k}.\end{aligned}$$

So at time  $t = 1$  the velocity of the particle is

$$\mathbf{v} = 4 \cos 4 \mathbf{i} - 4 \sin 4 \mathbf{j} + 3 \mathbf{k},$$

and the speed is

$$\begin{aligned}|\mathbf{v}| &= \sqrt{16 \cos^2 4 + 16 \sin^2 4 + 3^2} \\ &= \sqrt{16 + 9} = \sqrt{25} = 5.\end{aligned}$$

### Solution to Activity 21

The position of the particle in terms of time  $t$  is

$$\begin{aligned}\mathbf{r} &= \int \mathbf{v} dt = \int (4t \mathbf{i} + 3t^2 \mathbf{j}) dt \\ &= 2t^2 \mathbf{i} + t^3 \mathbf{j} + \mathbf{c},\end{aligned}$$

where  $\mathbf{c}$  is a vector constant.

At time  $t = 1$ , the position is  $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j}$ .

Substituting these values into the equation above gives

$$2\mathbf{i} + 3\mathbf{j} = 2\mathbf{i} + \mathbf{j} + \mathbf{c}.$$

Hence  $\mathbf{c} = 2\mathbf{j}$ , and therefore

$$\mathbf{r} = 2t^2 \mathbf{i} + (t^3 + 2) \mathbf{j}.$$

At time  $t = 2$ , the particle has position

$$\begin{aligned}\mathbf{r} &= (2 \times 2^2) \mathbf{i} + (2^3 + 2) \mathbf{j} \\ &= 8\mathbf{i} + 10\mathbf{j},\end{aligned}$$

and its distance from the origin is

$$|\mathbf{r}| = \sqrt{8^2 + 10^2} = 12.81 \text{ (to 2 d.p.)}.$$

### Solution to Activity 22

The velocity is given by

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} \\ &= \frac{d}{dt}(\sin t) \mathbf{i} - \frac{d}{dt}(\cos 2t) \mathbf{j} + \frac{d}{dt}(t) \mathbf{k} \\ &= \cos t \mathbf{i} - 2 \sin 2t \mathbf{j} + \mathbf{k}.\end{aligned}$$

The acceleration is given by

$$\begin{aligned}\mathbf{a} &= \frac{d\mathbf{v}}{dt} \\ &= \frac{d}{dt}(\cos t) \mathbf{i} - \frac{d}{dt}(2 \sin 2t) \mathbf{j} + \frac{d}{dt}(1) \mathbf{k} \\ &= -\sin t \mathbf{i} - 4 \cos 2t \mathbf{j}.\end{aligned}$$

### Solution to Activity 23

The velocity is given by

$$\begin{aligned}\mathbf{v} &= \int \mathbf{a} dt \\ &= \left(4 \int e^{2t} dt\right) \mathbf{i} - \left(\int 2 dt\right) \mathbf{j} \\ &= 2e^{2t} \mathbf{i} - 2t \mathbf{j} + \mathbf{c},\end{aligned}$$

where  $\mathbf{c}$  is a vector constant.

We know that  $\mathbf{v} = \mathbf{i}$  when  $t = 0$ . Substituting these values into the equation above gives

$$\mathbf{i} = 2\mathbf{i} + \mathbf{c},$$

so  $\mathbf{c} = -\mathbf{i}$ . This gives

$$\begin{aligned}\mathbf{v} &= 2e^{2t} \mathbf{i} - 2t \mathbf{j} - \mathbf{i} \\ &= (2e^{2t} - 1) \mathbf{i} - 2t \mathbf{j}.\end{aligned}$$

Hence the position is given by

$$\begin{aligned}\mathbf{r} &= \int \mathbf{v} dt \\ &= \int ((2e^{2t} - 1) \mathbf{i} - 2t \mathbf{j}) dt \\ &= (e^{2t} - t) \mathbf{i} - t^2 \mathbf{j} + \mathbf{d},\end{aligned}$$

where  $\mathbf{d}$  is a vector constant.

We know that  $\mathbf{r} = 2\mathbf{j}$  when  $t = 0$ . Substituting these values into the equation above gives

$$2\mathbf{j} = \mathbf{i} + \mathbf{d},$$

so  $\mathbf{d} = -\mathbf{i} + 2\mathbf{j}$ . This gives

$$\begin{aligned}\mathbf{r} &= (e^{2t} - t) \mathbf{i} - t^2 \mathbf{j} - \mathbf{i} + 2\mathbf{j} \\ &= (e^{2t} - t - 1) \mathbf{i} + (2 - t^2) \mathbf{j}.\end{aligned}$$

### Solution to Activity 24

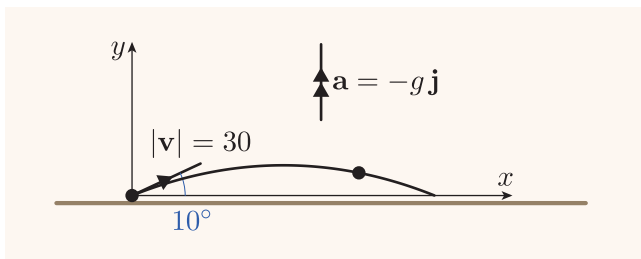
For each launch speed, the projectile seems to have the greatest range when it is launched at an angle of  $45^\circ$ .

### Solution to Activity 25

Model the ball as a particle.

Measure time from the moment that the ball is kicked.

Choose the  $x$ -axis to be horizontal, directly underneath the trajectory of the ball, pointing in the direction in which the ball travels horizontally. Choose the  $y$ -axis to point vertically upwards. Choose the origin to be at the point from where the ball is kicked.



We know that at time  $t = 0$  the position is  $\mathbf{r} = \mathbf{0}$  and the velocity is  $\mathbf{v} = 30 \cos 10^\circ \mathbf{i} + 30 \sin 10^\circ \mathbf{j}$ .

The acceleration  $\mathbf{a}$  is the acceleration due to gravity, so it has magnitude  $g$  and points downwards. This gives  $\mathbf{a} = -g\mathbf{j}$ .

The velocity of the ball is given by

$$\begin{aligned}\mathbf{v} &= \int \mathbf{a} dt \\ &= - \int g\mathbf{j} dt \\ &= -gt\mathbf{j} + \mathbf{c},\end{aligned}$$

where  $\mathbf{c}$  is a vector constant.

At time  $t = 0$  the velocity is

$$\mathbf{v} = 30 \cos 10^\circ \mathbf{i} + 30 \sin 10^\circ \mathbf{j}.$$

Substituting these values into the equation above gives

$$30 \cos 10^\circ \mathbf{i} + 30 \sin 10^\circ \mathbf{j} = -g \times 0 \mathbf{j} + \mathbf{c},$$

so

$$\mathbf{c} = 30 \cos 10^\circ \mathbf{i} + 30 \sin 10^\circ \mathbf{j}.$$

This gives

$$\begin{aligned}\mathbf{v} &= -gt\mathbf{j} + 30 \cos 10^\circ \mathbf{i} + 30 \sin 10^\circ \mathbf{j} \\ &= 30 \cos 10^\circ \mathbf{i} + (30 \sin 10^\circ - gt)\mathbf{j}.\end{aligned}$$

The position of the ball is given by

$$\begin{aligned}\mathbf{r} &= \int \mathbf{v} dt \\ &= \int (30 \cos 10^\circ \mathbf{i} + (30 \sin 10^\circ - gt)\mathbf{j}) dt \\ &= 30t \cos 10^\circ \mathbf{i} + (30t \sin 10^\circ - \frac{1}{2}gt^2)\mathbf{j} + \mathbf{d},\end{aligned}$$

where  $\mathbf{d}$  is a vector constant.

At time  $t = 0$  the position is  $\mathbf{r} = \mathbf{0}$ . Substituting these values into the equation above gives

$$\mathbf{0} = 30 \times 0 \cos 10^\circ \mathbf{i} + (30 \times 0 \sin 10^\circ - \frac{1}{2}g \times 0^2)\mathbf{j} + \mathbf{d},$$

so  $\mathbf{d} = \mathbf{0}$ .

Hence the position of the ball at time  $t$  is given by

$$\mathbf{r} = 30t \cos 10^\circ \mathbf{i} + (30t \sin 10^\circ - \frac{1}{2}gt^2)\mathbf{j}.$$

- (a) The ball hits the ground when the  $\mathbf{j}$ -component of its position is 0; that is, when

$$30t \sin 10^\circ - \frac{1}{2}gt^2 = 0.$$

Factorising gives

$$t(30 \sin 10^\circ - \frac{1}{2}gt) = 0.$$

This equation has two solutions, namely  $t = 0$  and the solution given by

$$30 \sin 10^\circ - \frac{1}{2}gt = 0.$$

However the time  $t$  when the ball hits the ground will be positive, so the second solution is the one required. Solving the equation above gives

$$t = \frac{60 \sin 10^\circ}{g}.$$

At this time  $t$ , the  $\mathbf{i}$ -component of the position is

$$\begin{aligned}30t \cos 10^\circ &= \frac{30 \times 60 \sin 10^\circ \cos 10^\circ}{g} \\ &= \frac{1800 \sin 10^\circ \cos 10^\circ}{9.8} \\ &= 31.4 \dots\end{aligned}$$

So the ball travels a horizontal distance of 31 m (to 2 s.f.) before it hits the ground.

- (b) The ball reaches its highest position when the  $\mathbf{j}$ -component of its velocity is 0; that is, when

$$30 \sin 10^\circ - gt = 0.$$

This gives

$$t = \frac{30 \sin 10^\circ}{g}.$$

At this time, the  $\mathbf{j}$ -component of its position is

$$\begin{aligned}30t \sin 10^\circ - \frac{1}{2}gt^2 &= 30 \left( \frac{30 \sin 10^\circ}{g} \right) \sin 10^\circ - \frac{1}{2}g \left( \frac{30 \sin 10^\circ}{g} \right)^2 \\ &= \frac{30^2 \sin^2 10^\circ}{g} - \frac{30^2 \sin^2 10^\circ}{2g} \\ &= \frac{30^2 \sin^2 10^\circ}{2g} \\ &= \frac{900 \sin^2 10^\circ}{2 \times 9.8} \\ &= 1.38 \dots\end{aligned}$$

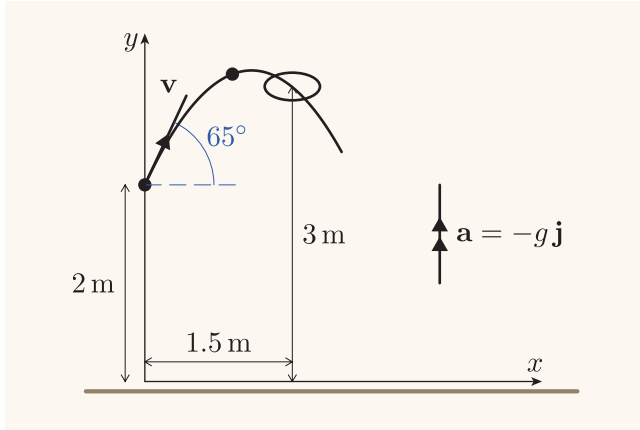
So the ball reaches a maximum height of 1.4 m (to 2 s.f.).

## Solution to Activity 26

Model the ball as a particle.

Measure time from the moment the ball is thrown.

Choose the  $x$ -axis to be horizontal, pointing in the direction from the player to the hoop, and choose the  $y$ -axis to point vertically upwards, with the origin at the point on the ground vertically below the point from where the ball is thrown.



Let the initial speed of the ball be  $v_0$ .

We know that at time  $t = 0$  the position is  $\mathbf{r} = 2\mathbf{j}$  and the velocity is

$$\mathbf{v} = v_0 \cos 65^\circ \mathbf{i} + v_0 \sin 65^\circ \mathbf{j}.$$

The position of the hoop is  $1.5\mathbf{i} + 3\mathbf{j}$ .

The acceleration  $\mathbf{a}$  is the acceleration due to gravity, so it has magnitude  $g$  and points downwards. This gives  $\mathbf{a} = -g\mathbf{j}$ .

The velocity of the ball is given by

$$\begin{aligned}\mathbf{v} &= \int \mathbf{a} \, dt \\ &= - \int g\mathbf{j} \, dt \\ &= -gt\mathbf{j} + \mathbf{c},\end{aligned}$$

where  $\mathbf{c}$  is a vector constant.

At time  $t = 0$ , the velocity is

$$\mathbf{v} = v_0 \cos 65^\circ \mathbf{i} + v_0 \sin 65^\circ \mathbf{j}.$$

Substituting these values into the equation above gives

$$v_0 \cos 65^\circ \mathbf{i} + v_0 \sin 65^\circ \mathbf{j} = -g \times 0 \mathbf{j} + \mathbf{c},$$

which gives

$$\mathbf{c} = v_0 \cos 65^\circ \mathbf{i} + v_0 \sin 65^\circ \mathbf{j}.$$

This gives

$$\begin{aligned}\mathbf{v} &= -gt\mathbf{j} + v_0 \cos 65^\circ \mathbf{i} + v_0 \sin 65^\circ \mathbf{j} \\ &= v_0 \cos 65^\circ \mathbf{i} + (v_0 \sin 65^\circ - gt)\mathbf{j}.\end{aligned}$$

The position of the ball is given by

$$\begin{aligned}\mathbf{r} &= \int \mathbf{v} \, dt \\ &= \int (v_0 \cos 65^\circ \mathbf{i} + (v_0 \sin 65^\circ - gt)\mathbf{j}) \, dt \\ &= v_0 t \cos 65^\circ \mathbf{i} + (v_0 t \sin 65^\circ - \tfrac{1}{2}gt^2)\mathbf{j} + \mathbf{d},\end{aligned}$$

where  $\mathbf{d}$  is a vector constant.

At time  $t = 0$  the position is  $\mathbf{r} = 2\mathbf{j}$ . Substituting these values into the equation above gives

$$\begin{aligned}2\mathbf{j} &= v_0 \times 0 \times \cos 65^\circ \mathbf{i} \\ &\quad + (v_0 \times 0 \times \sin 65^\circ - \tfrac{1}{2}g \times 0^2)\mathbf{j} + \mathbf{d},\end{aligned}$$

so  $\mathbf{d} = 2\mathbf{j}$ . This gives

$$\begin{aligned}\mathbf{r} &= v_0 t \cos 65^\circ \mathbf{i} + (v_0 t \sin 65^\circ - \tfrac{1}{2}gt^2)\mathbf{j} + 2\mathbf{j} \\ &= v_0 t \cos 65^\circ \mathbf{i} + (v_0 t \sin 65^\circ - \tfrac{1}{2}gt^2 + 2)\mathbf{j}.\end{aligned}$$

For the centre of the ball to go through the centre of the hoop, which is at position  $1.5\mathbf{i} + 3\mathbf{j}$ , we need

$$\mathbf{r} = 1.5\mathbf{i} + 3\mathbf{j}$$

at some time  $t$ .

This equation gives

$$v_0 t \cos 65^\circ \mathbf{i} + (v_0 t \sin 65^\circ - \tfrac{1}{2}gt^2 + 2)\mathbf{j} = 1.5\mathbf{i} + 3\mathbf{j}.$$

Resolving this vector equation in the  $\mathbf{i}$ - and  $\mathbf{j}$ -directions gives

$$v_0 t \cos 65^\circ = 1.5$$

$$v_0 t \sin 65^\circ - \tfrac{1}{2}gt^2 + 2 = 3.$$

These are simultaneous equations in  $v_0$  and  $t$ . The first equation gives

$$t = \frac{1.5}{v_0 \cos 65^\circ},$$

and using this equation to substitute for  $t$  in the second equation gives

$$\frac{1.5v_0 \sin 65^\circ}{v_0 \cos 65^\circ} - \tfrac{1}{2}g \left( \frac{1.5}{v_0 \cos 65^\circ} \right)^2 + 2 = 3.$$

Simplifying gives

$$1.5 \tan 65^\circ - \frac{2.25g}{2v_0^2 \cos^2 65^\circ} = 1$$

so

$$\frac{1.125g}{v_0^2 \cos^2 65^\circ} = 1.5 \tan 65^\circ - 1,$$

which gives

$$v_0^2 = \frac{1.125g}{\cos^2 65^\circ (1.5 \tan 65^\circ - 1)}.$$

Since  $v_0$  is positive, we obtain

$$v_0 = \sqrt{\frac{1.125g}{\cos^2 65^\circ (1.5 \tan 65^\circ - 1)}}.$$

Evaluating this expression, taking  $g = 9.8 \text{ m s}^{-2}$ , gives

$$v_0 = 5.27 \dots$$

So the player needs to throw the ball at a speed of  $5.3 \text{ m s}^{-1}$  (to 2 s.f.).

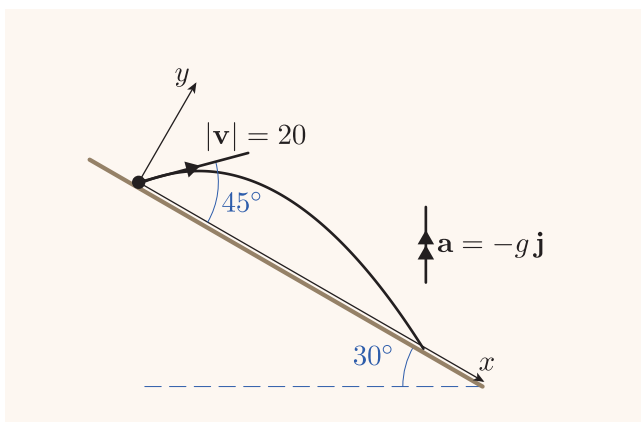
(In solving this problem you might have chosen the origin to be at the point from where the ball is thrown, rather than at the point on the ground vertically below the point from where the ball is thrown. This makes the working very slightly easier, as the vector constant  $\mathbf{d}$  is then  $\mathbf{0}$ , but you need to be careful to use the height of the hoop above the point from which the ball was thrown, rather than its height above the ground.)

### Solution to Activity 27

Model the snowboarder as a particle.

Measure time from the moment the snowboarder leaves the ground.

Choose the  $x$ -axis to point down the slope, and choose the  $y$ -axis to be perpendicular to the slope, with the origin at the point from which the snowboarder takes off.



We know that at time  $t = 0$  the position is  $\mathbf{r} = \mathbf{0}$  and the velocity is

$$\begin{aligned} \mathbf{v} &= 20 \cos 45^\circ \mathbf{i} + 20 \sin 45^\circ \mathbf{j} \\ &= 10\sqrt{2} \mathbf{i} + 10\sqrt{2} \mathbf{j}. \end{aligned}$$

The acceleration  $\mathbf{a}$  is the acceleration due to gravity, so it has magnitude  $g$  and points downwards.



Expressing  $\mathbf{a}$  in component form gives

$$\begin{aligned} \mathbf{a} &= g \cos 60^\circ \mathbf{i} - g \sin 60^\circ \mathbf{j} \\ &= \frac{1}{2}g \mathbf{i} - \frac{\sqrt{3}}{2}g \mathbf{j}. \end{aligned}$$

The velocity of the snowboarder is given by

$$\begin{aligned} \mathbf{v} &= \int \mathbf{a} \, dt \\ &= \int \left( \frac{1}{2}g \mathbf{i} - \frac{\sqrt{3}}{2}g \mathbf{j} \right) dt \\ &= \frac{1}{2}gt \mathbf{i} - \frac{\sqrt{3}}{2}gt \mathbf{j} + \mathbf{c}, \end{aligned}$$

where  $\mathbf{c}$  is a vector constant.

We know that at time  $t = 0$  the velocity is

$$\mathbf{v} = 10\sqrt{2} \mathbf{i} + 10\sqrt{2} \mathbf{j}.$$

Substituting these values into the equation above gives

$$10\sqrt{2} \mathbf{i} + 10\sqrt{2} \mathbf{j} = \frac{1}{2}g \times 0 \mathbf{i} - \frac{\sqrt{3}}{2}g \times 0 \mathbf{j} + \mathbf{c},$$

so

$$\mathbf{c} = 10\sqrt{2} \mathbf{i} + 10\sqrt{2} \mathbf{j}.$$

This gives

$$\begin{aligned} \mathbf{v} &= \frac{1}{2}gt \mathbf{i} - \frac{\sqrt{3}}{2}gt \mathbf{j} + 10\sqrt{2} \mathbf{i} + 10\sqrt{2} \mathbf{j} \\ &= \left( 10\sqrt{2} + \frac{1}{2}gt \right) \mathbf{i} + \left( 10\sqrt{2} - \frac{\sqrt{3}}{2}gt \right) \mathbf{j}. \end{aligned}$$

The position of the snowboarder is given by

$$\begin{aligned} \mathbf{r} &= \int \mathbf{v} \, dt \\ &= \int \left( \left( 10\sqrt{2} + \frac{1}{2}gt \right) \mathbf{i} + \left( 10\sqrt{2} - \frac{\sqrt{3}}{2}gt \right) \mathbf{j} \right) dt \\ &= \left( 10\sqrt{2}t + \frac{1}{4}gt^2 \right) \mathbf{i} + \left( 10\sqrt{2}t - \frac{\sqrt{3}}{4}gt^2 \right) \mathbf{j} + \mathbf{d}, \end{aligned}$$

where  $\mathbf{d}$  is a vector constant.

## Unit 10 Dynamics

We know that when  $t = 0$  the position is  $\mathbf{r} = \mathbf{0}$ . Substituting these values into the equation above gives

$$\mathbf{0} = \left( 10\sqrt{2} \times 0 + \frac{1}{4}g \times 0^2 \right) \mathbf{i} + \left( 10\sqrt{2} \times 0 - \frac{\sqrt{3}}{4}g \times 0^2 \right) \mathbf{j} + \mathbf{d},$$

so  $\mathbf{d} = \mathbf{0}$ .

This gives

$$\mathbf{r} = \left( 10\sqrt{2}t + \frac{1}{4}gt^2 \right) \mathbf{i} + \left( 10\sqrt{2}t - \frac{\sqrt{3}}{4}gt^2 \right) \mathbf{j}.$$

The snowboarder lands back on the slope when the  $\mathbf{j}$ -component of his position is zero; that is, when

$$10\sqrt{2}t - \frac{\sqrt{3}}{4}gt^2 = 0.$$

Factorising this equation gives

$$t \left( 10\sqrt{2} - \frac{\sqrt{3}}{4}gt \right) = 0.$$

This equation has two solutions, namely  $t = 0$  (which is the moment of take-off) and the solution  $t$  given by

$$10\sqrt{2} - \frac{\sqrt{3}}{4}gt = 0.$$

Rearranging this equation gives

$$\frac{\sqrt{3}}{4}gt = 10\sqrt{2}$$

so

$$\begin{aligned} t &= 10\sqrt{2} \times \frac{4}{g\sqrt{3}} \\ &= \frac{40\sqrt{2}}{9.8\sqrt{3}} \\ &= 3.33 \dots \end{aligned}$$

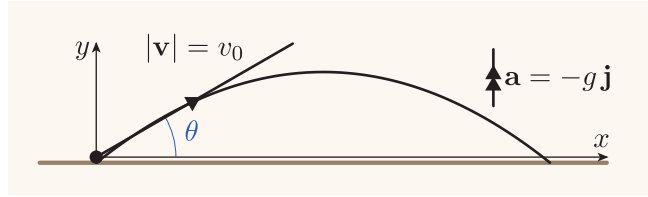
So the snowboarder spends 3.3 seconds (to 2 s.f.) in the air.

### Solution to Activity 28

(a) Model the projectile as a particle.

Measure time from the moment the projectile is launched.

Choose the  $x$ -axis to be horizontal, pointing along the ground in the direction of travel of the projectile, and choose the  $y$ -axis to point vertically upwards, with the origin at the point from which the projectile is launched.



We know that at time  $t = 0$  the position of the projectile is  $\mathbf{r} = \mathbf{0}$ . Also at time  $t = 0$  the velocity has magnitude  $v_0$  and is directed at the angle  $\theta$  above the horizontal, so in component form it is

$$\mathbf{v} = v_0 \cos \theta \mathbf{i} + v_0 \sin \theta \mathbf{j}.$$

The acceleration  $\mathbf{a}$  is the acceleration due to gravity, so it has magnitude  $g$  and points downwards. This gives  $\mathbf{a} = -g\mathbf{j}$ .

The velocity of the projectile is given by

$$\begin{aligned} \mathbf{v} &= \int \mathbf{a} \, dt \\ &= - \int g \mathbf{j} \, dt \\ &= -gt \mathbf{j} + \mathbf{c}, \end{aligned}$$

where  $\mathbf{c}$  is a vector constant.

At time  $t = 0$  the velocity is

$$\mathbf{v} = v_0 \cos \theta \mathbf{i} + v_0 \sin \theta \mathbf{j}.$$

Substituting these values into the equation above gives

$$v_0 \cos \theta \mathbf{i} + v_0 \sin \theta \mathbf{j} = -g \times 0 \mathbf{j} + \mathbf{c},$$

so

$$\mathbf{c} = v_0 \cos \theta \mathbf{i} + v_0 \sin \theta \mathbf{j}.$$

This gives

$$\begin{aligned} \mathbf{v} &= -gt \mathbf{j} + v_0 \cos \theta \mathbf{i} + v_0 \sin \theta \mathbf{j} \\ &= v_0 \cos \theta \mathbf{i} + (v_0 \sin \theta - gt) \mathbf{j}. \end{aligned}$$

The position of the projectile is given by

$$\begin{aligned} \mathbf{r} &= \int \mathbf{v} \, dt \\ &= \int (v_0 \cos \theta \mathbf{i} + (v_0 \sin \theta - gt) \mathbf{j}) \, dt \\ &= v_0 t \cos \theta \mathbf{i} + (v_0 t \sin \theta - \frac{1}{2}gt^2) \mathbf{j} + \mathbf{d}, \end{aligned}$$

where  $\mathbf{d}$  is a vector constant.

At time  $t = 0$  the position is  $\mathbf{r} = \mathbf{0}$ . Substituting these values into the equation above gives

$$\begin{aligned} \mathbf{0} &= v_0 \times 0 \times \cos \theta \mathbf{i} \\ &\quad + (v_0 \times 0 \times \sin \theta - \frac{1}{2}g \times 0^2) \mathbf{j} + \mathbf{d}, \end{aligned}$$

so  $\mathbf{d} = \mathbf{0}$ .



This gives

$$\mathbf{r} = v_0 t \cos \theta \mathbf{i} + (v_0 t \sin \theta - \frac{1}{2}gt^2) \mathbf{j}.$$

The projectile returns to the ground when the  $\mathbf{j}$ -component of its position is zero; that is, when

$$v_0 t \sin \theta - \frac{1}{2}gt^2 = 0.$$

Factorising this equation gives

$$t(v_0 \sin \theta - \frac{1}{2}gt) = 0.$$

This equation gives two solutions for  $t$ , namely  $t = 0$  (which gives the time at which the projectile is launched), and

$$t = \frac{2v_0 \sin \theta}{g},$$

which gives the time at which the projectile returns to the ground.

The range of the projectile (the horizontal distance that it travels) is given by the  $\mathbf{i}$ -component of its position at the time when it returns to the ground. This is

$$\begin{aligned} v_0 t \cos \theta &= v_0 \frac{2v_0 \sin \theta}{g} \cos \theta \\ &= \frac{2v_0^2 \sin \theta \cos \theta}{g}. \end{aligned}$$

So the range of the projectile is

$$\frac{2v_0^2 \sin \theta \cos \theta}{g},$$

as required.

(b) Using the trigonometric identity

$$\sin 2\theta = 2 \sin \theta \cos \theta,$$

we can write the expression found in part (a) for the range of the projectile as

$$\frac{v_0^2 \sin 2\theta}{g}.$$

The maximum value of  $\sin 2\theta$  is 1, and, since  $0 \leq \theta \leq 90^\circ$ , which gives  $0 \leq 2\theta \leq 180^\circ$ , this is achieved when  $2\theta = 90^\circ$ ; that is, when  $\theta = 45^\circ$ .

So the maximum range of the projectile is given by  $v_0^2/g$ , and this is achieved when  $\theta = 45^\circ$ .

We have found that no matter what the initial speed  $v_0$  of the projectile is, the maximum range over horizontal ground is achieved when the launch angle is  $45^\circ$ .

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